

**The Vanishing Moment Method for Fully
Nonlinear Second Order Partial Differential
Equations: Formulation, Theory, and Numerical
Analysis**

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Abstract

The vanishing moment method was introduced by the authors in [37] as a reliable methodology for computing viscosity solutions of fully nonlinear second order partial differential equations (PDEs), in particular, using Galerkin-type numerical methods such as finite element methods, spectral methods, and discontinuous Galerkin methods, a task which has not been practicable in the past. The crux of the vanishing moment method is the simple idea of approximating a fully nonlinear second order PDE by a family (parametrized by a small parameter ε) of quasilinear higher order (in particular, fourth order) PDEs. The primary objectives of this book are to present a detailed convergent analysis for the method in the radial symmetric case and to carry out a comprehensive finite element numerical analysis for the vanishing moment equations (i.e., the regularized fourth order PDEs). Abstract methodological and convergence analysis frameworks of conforming finite element methods and mixed finite element methods are first developed for fully nonlinear second order PDEs in general settings. The abstract frameworks are then applied to three prototypical nonlinear equations, namely, the Monge-Ampère equation, the equation of prescribed Gauss curvature, and the infinity-Laplacian equation. Numerical experiments are also presented for each problem to validate the theoretical error estimate results and to gauge the efficiency of the proposed numerical methods and the vanishing moment methodology.

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CHAPTER 1

Prelude

1.1. Introduction

Fully nonlinear partial differential equations (PDEs) are those equations which are nonlinear in the highest order derivative(s) of the unknown function(s). In the case of the second order equations, the general form of fully nonlinear PDEs is given by

$$(1.1) \quad F(D^2u, \nabla u, u, x) = 0,$$

where $D^2u(x)$ and $\nabla u(x)$ denote respectively the Hessian and the gradient of u at $x \in \Omega \subset \mathbf{R}^n$. Here, F is assumed to be a nonlinear function in at least one of its entries of D^2u . Fully nonlinear PDEs arise from many scientific and engineering fields including differential geometry, optimal control, mass transportation, geostrophic fluid, meteorology, and general relativity (cf. [18, 19, 42, 41, 58] and the references therein).

Examples of such equations include (cf. [42])

- *The Monge-Ampère equation*

$$(1.2) \quad \det(D^2u) = f.$$

- *The equation of prescribed Gauss curvature*

$$(1.3) \quad \det(D^2u) = \mathcal{K}(1 + |\nabla u|^2)^{\frac{n+2}{2}}.$$

- *The Bellman equation*

$$(1.4) \quad \inf_{\theta \in V} (L_\theta u - f_\theta) = 0.$$

Here, $\det(D^2u(x))$ denotes the determinant of the Hessian D^2u at x , and $\{L_\theta\}$ denotes a family of second order linear differential operators.

Because of the full nonlinearity in (1.1), the standard weak solution theory based on the integration by parts approach does not work and other notions of weak solutions must be sought. Progress has been made in the latter half of the twentieth century concerning this issue after the introduction of viscosity solutions. In 1983, Crandall and Lions [24] introduced the notion of viscosity solutions and used the vanishing viscosity method to show existence of a solution for the Hamilton-Jacobi equation:

$$(1.5) \quad u_t + H(\nabla u, u, x) = 0 \quad \text{in } \mathbf{R}^n \times (0, \infty).$$

The vanishing viscosity method approximates the Hamilton-Jacobi equation by the following regularized, second order quasilinear PDE:

$$(1.6) \quad u_t^\varepsilon - \varepsilon \Delta u^\varepsilon + H(\nabla u^\varepsilon, u^\varepsilon, x) = 0 \quad \text{in } \mathbf{R}^n \times (0, \infty).$$

It was shown that [24] there exists a unique solution u^ε to the regularized Cauchy problem that converges locally and uniformly to a continuous function u which is defined to be a viscosity solution of the Hamilton-Jacobi equation (1.5). However, to establish uniqueness, the following intrinsic definition of viscosity solutions was also proposed [24, 25]:

Definition 1.1. Let $H : \mathbf{R}^n \times \mathbf{R} \times \Omega \rightarrow \mathbf{R}$ and $g : \partial\Omega \rightarrow \mathbf{R}$ be continuous functions, and consider the following problem:

$$(1.7) \quad H(\nabla u, u, x) = 0 \quad \text{in } \Omega,$$

$$(1.8) \quad u = g \quad \text{on } \partial\Omega.$$

- (i) $u \in C^0(\Omega)$ is called a *viscosity subsolution* of (1.7)–(1.8) if $u|_{\partial\Omega} = g$, and for every C^1 function $\varphi(x)$ such that $u - \varphi$ has a local maximum at $x_0 \in \Omega$, there holds

$$H(\nabla\varphi(x_0), u(x_0), x_0) \leq 0.$$

- (ii) $u \in C^0(\Omega)$ is called a *viscosity supersolution* of (1.7)–(1.8) if $u|_{\partial\Omega} = g$, and for every C^1 function $\varphi(x)$ such that $u - \varphi$ has a local minimum at $x_0 \in \Omega$, there holds

$$H(\nabla\varphi(x_0), u(x_0), x_0) \geq 0.$$

- (iii) $u \in C^0(\Omega)$ is called a *viscosity solution* of (1.7)–(1.8) if it is both a viscosity subsolution and supersolution.

Clearly, the above definition is not variational, as it is based on a “differentiation by parts” approach (a terminology introduced in [24, 25]). In addition, the word “viscosity” loses its original meaning in the definition. However, it was shown [24, 25] that every viscosity solution constructed by the vanishing viscosity method is an intrinsic viscosity solution (i.e., a solution that satisfies Definition 1.1). Besides addressing the uniqueness issue, another reason to favor the intrinsic differentiation by parts definition is that the definition and the notion of viscosity solutions can be readily extended to fully nonlinear second order PDEs as follows (cf. [18]):

Definition 1.2. Let $F : \mathbf{R}^{n \times n} \times \mathbf{R}^n \times \mathbf{R} \times \Omega \rightarrow \mathbf{R}$ and $g : \partial\Omega \rightarrow \mathbf{R}$ be continuous functions, and consider the following problem:

$$(1.9) \quad F(D^2u, \nabla u, u, x) = 0 \quad \text{in } \Omega,$$

$$(1.10) \quad u = g \quad \text{on } \partial\Omega.$$

- (i) $u \in C^0(\Omega)$ is called a *viscosity subsolution* of (1.9)–(1.10) if $u|_{\partial\Omega} = g$, and for every C^2 function $\varphi(x)$ such that $u - \varphi$ has a local maximum at $x_0 \in \Omega$, there holds

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \leq 0.$$

- (ii) $u \in C^0(\Omega)$ is called a *viscosity supersolution* of (1.9)–(1.10) if $u|_{\partial\Omega} = g$, and for every C^2 function $\varphi(x)$ such that $u - \varphi$ has a local minimum at $x_0 \in \Omega$, there holds

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \geq 0.$$

- (iii) $u \in C^0(\Omega)$ is called a *viscosity solution* of (1.9)–(1.10) if it is both a viscosity subsolution and supersolution.

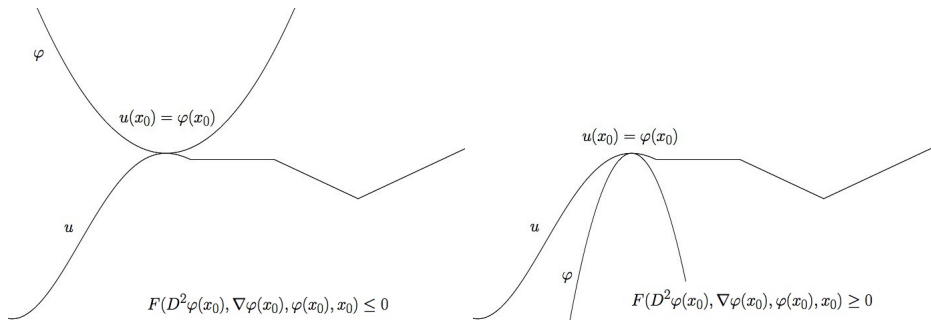


FIGURE 1. A Geometric interpretation of viscosity solutions

Remark 1.3. Without loss of generality, we may assume that $u(x_0) = \varphi(x_0)$ whenever $u - \varphi$ achieves a local maximum or local minimum at $x_0 \in \Omega$ in Definition 1.2. Therefore, in an informal setting, u is a viscosity solution if for every smooth function φ that “touches” the graph of u from above at x_0 (see Figure 1) there holds

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), \varphi(x_0), x_0) \leq 0,$$

and if φ “touches” the graph of u from below at x_0 , then

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), \varphi(x_0), x_0) \geq 0.$$

In case of the fully nonlinear *first order* PDEs, tremendous progress has been made in the past three decades in terms of PDE analysis and numerical methods. A profound viscosity solution theory has been established (cf. [24, 25, 26, 41]) and a wealth of efficient and robust numerical methods and algorithms have been developed and implemented (cf. [9, 14, 23, 27, 56, 66, 67, 68, 72, 79, 80]). However, in the case of fully nonlinear *second order* PDEs, the situation is strikingly different. On the one hand, there have been enormous advances in PDE analysis in the past two decades after the introduction of the notion of viscosity solutions by M. Crandall and P. L. Lions in 1983 (cf. [18, 26, 44]). On the other hand, in contrast to the success of the PDE analysis, numerical solutions for general fully nonlinear second order PDEs is a relatively untouched area.

There are several reasons for this lack of progress in numerical methods. First, the most obvious difficulty is the full nonlinearity in the equation. Second, solutions to fully nonlinear second order equations are often only unique in a certain class of functions, and this conditional uniqueness is very difficult to handle numerically. Lastly and most importantly, it is extremely difficult (if all possible) to mimic the differentiation by parts approach at the discrete level. As a consequence, there is little hope to develop a discrete viscosity solution theory. Furthermore, it is impossible to directly compute viscosity solutions using Galerkin-type numerical methods including finite element methods, spectral Galerkin methods, and discontinuous Galerkin methods, since they are all based on variational formulations of PDEs. In fact, this is clear from the definition of viscosity solutions, which is not based on the traditional integration by parts approach, but rather is defined by the differentiation by parts approach.

To explain the above points, consider the Dirichlet problem for the Monge-Ampère equation as an example:

$$(1.11) \quad \det(D^2u) = f \quad \text{in } \Omega,$$

$$(1.12) \quad u = g \quad \text{on } \partial\Omega,$$

which corresponds to $F(D^2u, \nabla u, u, x) = f(x) - \det(D^2u)$. It is known that for a non-strictly convex domain Ω , the above problem does not have classical solutions in general even if f, g , and $\partial\Omega$ are smooth [42]. Results of A. D. Aleksandrov state that the Dirichlet problem with $f > 0$ has a unique generalized solution (which is also the viscosity solution) in the class of convex functions [2, 44]. The reason to restrict the admissible set to be the set of convex functions is that the Monge-Ampère equation is only elliptic in that set [42, 44]. It should be noted that in general, the Dirichlet problem (1.11)–(1.12) may have other nonconvex solutions even when $f > 0$. It is easy to see that if one discretizes (1.11) directly using a standard finite difference method, not only would the resulting algebraic system be difficult to solve, one immediately loses control on which solution the numerical scheme approximates - and this is assuming that the nonlinear discrete problem has solutions! Furthermore, the situation is even worse if one tries to formulate a Galerkin-type numerical method because there is not a variational or weak formulation in which to start.

Nevertheless, a few recent numerical attempts and results have been known in the literature. In [65] Olier and Prussner proposed a finite difference scheme for computing Aleksandrov measure induced by D^2u (and obtained the solution u of (1.2) as a by-product) in two dimensions. The scheme is extremely geometric and difficult to generalize to other fully nonlinear second order PDEs. In [6] Barles and Souganidis showed that any monotone, stable, and consistent finite difference scheme converges to the viscosity solution provided that there exists a comparison principle for the limiting equation. Their result provides a guideline for constructing convergent finite difference methods, but they did not address how to construct such a scheme. In [4], Baginski and Whitaker proposed a finite difference scheme for the equation of prescribed Gauss curvature (1.3) in two dimensions by mimicking the unique continuation method (used to prove existence of the PDE) at the discrete level. The method becomes very unstable when the homotopy is dominated by the fully nonlinear equation. Oberman [64] constructed a wide stencil finite difference scheme for fully nonlinear elliptic PDEs which can be written as functions of eigenvalues of the Hessian matrix and proved that the scheme satisfies the convergence criterion established by Barles and Souganidis in [6]. In a series of papers [29, 30, 31] Dean and Glowinski proposed an augmented Lagrange multiplier method and a least squares method for problem (1.2) and Pucci's equation (cf. [18, 42]) in two dimensions by treating the nonlinear PDEs as a constraint and using a variational criterion to select a particular solution. However, as the admissible set is contained in $H^2(\Omega)$, it could become empty if all solutions of the underlying fully nonlinear PDE are not differentiable. Finally, Böhmer [15] recently introduced a projection method using C^1 finite elements for approximating classical solutions of a certain class of fully nonlinear second order elliptic PDEs. However, the issue of how to reliably compute a selected solution (the resulting discrete problem often has multiple solutions) was not addressed and still remains

an open question. Numerical experiments were reported in [65, 4, 64, 29, 30, 31], however, convergence analysis was not addressed except in [64].

In addition, we like to remark that there is a considerable amount of literature available on using finite difference methods to approximate viscosity solutions of fully nonlinear second order Bellman-type PDEs arising from stochastic optimal control (cf. [6, 7, 50, 53]). However, due to the special structure of Bellman-type PDEs, the approach used and the methods proposed in those papers could not be extended to other types of fully nonlinear second order PDEs since the construction of those methods critically relies on the linearity of the operators L_θ .

The first goal of this book is to present a general framework for the vanishing moment method and the notion of moment solutions for fully nonlinear second order PDEs. The vanishing moment method is very much in the same spirit of the vanishing viscosity method introduced in [24], and the notion of moment solutions for fully nonlinear second order PDEs is a natural extension of the (original) notion of viscosity solutions for fully nonlinear first order PDEs. This methodology was first introduced by the authors in [37] as a reliable way for computing viscosity solutions of fully nonlinear second order PDEs, in particular, using Galerkin-type numerical methods. The crux of this new method is to approximate a fully nonlinear second order PDE by a family of quasilinear fourth order PDEs. The limit of the solutions of the fourth order PDEs (if it exists) is defined as a moment solution of the original fully nonlinear second order PDE. As moment solutions are defined constructively, they can be readily computed by existing numerical methods. In the case of Monge-Ampère-type equations, extensive numerical experiments in [37, 38, 39, 62] suggest that the moment solution coincides with the viscosity solution as long as the latter exists. In this book, we shall present a detailed convergence theory for the vanishing moment method in the radial symmetric case. This then provides a theoretical foundation for the method and for the numerical results of [37, 38, 39, 62].

The second goal of this book, which is the bulk of the book's content, is to carry out a comprehensive finite element numerical analysis for the vanishing moment method. Two abstract frameworks are developed for this purpose in a general setting. The first framework concerns C^1 conforming finite element approximations of the vanishing moment equations (i.e., the regularized fourth order equations). The second framework develops (Herman-Miyoshi) mixed finite element methods for the vanishing moment equations. Each of these two frameworks consists of the formulation of the respective numerical methods, proving existence and uniqueness of numerical solutions, and deriving error estimates for the numerical solutions. Due to the strong nonlinearity of the PDEs, the standard numerical analysis techniques for finite element methods do not work here. To overcome the difficulty, we combine a fixed point argument with a linearization technique. After having completed both abstract frameworks, we apply them to three prototypical nonlinear equations, namely, the Monge-Ampère equation, the equation of prescribed Gauss curvature, and the infinity-Laplacian equation. The three equations are chosen because they present three different and interesting scenarios, that is, their linearizations are respectively coercive, indefinite, and degenerate. It is shown that our abstract frameworks are rich enough to cover all three scenarios.

The remainder of the book is organized as follows. Chapter 2 represents the formulation of the vanishing moment method and its informal insights. The material of this chapter has a large overlap with that of [37]. Chapter 3 is devoted to the convergence analysis of the vanishing moment method for the Monge-Ampère equation in the radial symmetric case. The main tasks of the chapter are to analyze the vanishing moment equations and to derive uniform (in ε) estimates for its solutions. The chapter also contains a convergence rate estimate result for the regularized solutions in the case that the viscosity solution of the Monge-Ampère equation belongs to $W^{2,\infty}(\Omega) \cap H^3(\Omega)$. Chapter 4 and 5 develop, respectively, the abstract frameworks for the two types of finite element (i.e., conforming and mixed finite element) approximations of the vanishing moment equations under some structure assumptions on the nonlinear differential operator F . Chapter 6 presents applications of the abstract frameworks of Chapter 4 and 5 to three prototypical nonlinear equations: the Monge-Ampère equation, the equation of prescribed Gauss curvature, and the infinity-Laplacian equation. For each equation, we formulate its vanishing moment approximations, subsequent finite element and mixed finite element methods, and obtain their error estimates by fitting the equation into the abstract frameworks. For the Monge-Ampère equation, besides some slight improvements, we essentially recover the early results reported in [38, 39]. On the other hand, the results for the equation of prescribed Gauss curvature and the infinity-Laplacian equation are new. In fact, to the best of our knowledge, no comparable results are known in the literature. Numerical experiments are also presented for each problem to validate the theoretical (error estimate) results, and to gauge the efficiency of the proposed numerical methods and the vanishing moment methodology. Finally, we end the book with a few concluding remarks in Chapter 7.

1.2. Preliminaries

Standard space notation is adopted in this book, we refer the reader to [13, 42, 22] for their exact definitions. In addition, Ω denotes a bounded convex domain in \mathbf{R}^n . (\cdot, \cdot) and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ denote the L^2 -inner products on Ω and on $\partial\Omega$, respectively. The unlabeled constant C is used to denote generic ε - and h -independent positive constants that may take on different values at different occurrences, where as labeled constants denote ε -dependent (but h -independent) constants. Furthermore all constants, labeled and unlabeled, are chapter-independent unless otherwise specified.

Throughout this book we assume that

$$F : \mathbf{R}^{n \times n} \times \mathbf{R}^n \times \mathbf{R} \times \Omega \longrightarrow \mathbf{R}$$

is a differentiable function in all its arguments. For a given (small) constant $\varepsilon > 0$, we define

$$G_\varepsilon(r, p, z, x) := \varepsilon \Delta \text{tr}(r) + F(r, p, z, x) \quad \forall r \in \mathbf{R}^{n \times n}, p \in \mathbf{R}^n, z \in \mathbf{R}, x \in \Omega.$$

For a given scalar function v and an $n \times n$ matrix-valued function $\mu = [\mu_{ij}]$ ¹, we set

$$\begin{aligned} F_r[r, p, z, x](\mu) &:= F_r : \mu = \sum_{i,j=1}^n \frac{\partial F}{\partial r_{ij}}(r, p, z, x) \mu_{ij}(x), \\ F_p[r, p, z, x](v) &:= F_p \cdot \nabla v = \sum_{i=1}^n \frac{\partial F}{\partial p_i}(r, p, z, x) \frac{\partial v}{\partial x_i}(x), \\ F_z[r, p, z, x](v) &:= F_z \cdot v = \frac{\partial F}{\partial z}(r, p, z, x) v(x), \\ F'[r, p, z, x](\mu, v) &:= F_r[r, p, z, x](\mu) + F_p[r, p, z, x](v) + F_z[r, p, z, x](v), \\ G'_\varepsilon[r, p, z, x](\mu, v) &:= \varepsilon \Delta \text{tr}(\mu) + F'[r, p, z, x](\mu, v). \end{aligned}$$

We also define, with a slight abuse of notation, for a scalar function w and an $n \times n$ tensor function $\kappa = [\kappa_{ij}]$, the following short-hand notation, which will be extensively used when developing mixed finite element methods in Chapter 5,

$$\begin{aligned} (1.13) \quad F(\kappa, w) &:= F(\kappa, \nabla w, w, x), \\ F_r[\kappa, w](\mu) &:= F_r[\kappa, \nabla w, w, x](\mu), \\ F_p[\kappa, w](v) &:= F_p[\kappa, \nabla w, w, x](v), \\ F_z[\kappa, w](v) &:= F_z[\kappa, \nabla w, w, x](v), \\ F'[\kappa, w](\mu, v) &:= F_r[\kappa, w](\mu) + F_p[\kappa, w](v) + F_z[\kappa, w](v), \\ G_\varepsilon(\kappa, w) &:= \varepsilon \Delta \text{tr}(\kappa) + F(\kappa, w), \\ G'_\varepsilon[\kappa, w](\mu, v) &:= \varepsilon \Delta \text{tr}(\mu) + F'[\kappa, w](\mu, v). \end{aligned}$$

For notation used in Chapter 4, we overload the operators F, G_ε, F' , and G'_ε once again and define the additional short-hand notation:

$$\begin{aligned} (1.14) \quad F(w) &:= F(D^2 w, w), \\ F_r[w](v) &:= F_r[D^2 w, w](D^2 v), \\ F_p[w](v) &:= F_p[D^2 w, w](v), \\ F_z[w](v) &:= F_z[D^2 w, w](v), \\ F'[w](v) &:= F_r[w](v) + F_p[w](v) + F_z[w](v), \\ G_\varepsilon(w) &:= G_\varepsilon(D^2 w, w) = \varepsilon \Delta^2 w + F(w), \\ G'_\varepsilon[w](v) &:= \varepsilon \Delta^2 v + F'[w](v). \end{aligned}$$

We conclude this section and chapter by citing a divergence-free row property of the cofactor matrix of the gradient of a vector-valued smooth function (a special case of Piola's identity). This property will be used many times in the later chapters of the book. A proof of this property can be found in [32, page 440].

Lemma 1.4. *Given a vector-valued function $\mathbf{v} = (v_1, v_2, \dots, v_n) : \Omega \rightarrow \mathbf{R}^n$. Assume $\mathbf{v} \in [C^2(\Omega)]^n$. Then the cofactor matrix $\text{cof}(D\mathbf{v})$ of the gradient matrix $D\mathbf{v}$*

¹In an effort to clarify notation, we mostly use Greek letters to represent matrix-valued functions, and Roman letters to represent scalar functions throughout the book

of \mathbf{v} satisfies the following row divergence-free property:

$$\operatorname{div}(\operatorname{cof}(D\mathbf{v}))_i = \sum_{j=1}^n \frac{\partial}{\partial x_j} (\operatorname{cof}(D\mathbf{v}))_{ij} = 0 \quad \text{for } i = 1, 2, \dots, n,$$

where $(\operatorname{cof}(D\mathbf{v}))_i$ and $(\operatorname{cof}(D\mathbf{v}))_{ij}$ denote respectively the i th row and the (i, j) -entry of $\operatorname{cof}(D\mathbf{v})$.

CHAPTER 2

Formulation of the vanishing moment method

In this chapter we shall present the formulation of the vanishing moment method for fully nonlinear second order PDE (1.1). We also explain how the method was conceived and give some informal insights about the method. We note that the material of this chapter has a large overlap with that of [37].

For the reasons and difficulties explained in Chapter 1, as far as we can see, it is unlikely (at least very difficult if at all possible) that one can directly approximate viscosity solutions of general fully nonlinear second order PDEs using available numerical methodologies such as finite difference methods, finite element methods, spectral and discontinuous Galerkin methods, meshless methods, etc. In particular, the robust and popular Galerkin-type methods (such as finite element methods, spectral, and discontinuous Galerkin methods) for solving linear and quasilinear PDEs become powerless when facing fully nonlinear second order PDEs. From a computational point of view, the notion of viscosity solutions is, in some sense, an “inconvenient” notion for fully nonlinear second order PDEs because it is neither constructive nor variational. In searching for a “better” notion of weak solutions for fully nonlinear second order PDEs, we are inspired by the following simple but crucial observation: *the crux of the vanishing viscosity method for the Hamilton-Jacobi equation and the original notion of viscosity solutions is to approximate a lower order fully nonlinear PDE by a family of quasilinear higher order PDEs.*

It is exactly this observation which motivates us to apply the above quoted idea to fully nonlinear second order PDE (1.1) in [37]. To this end, we take one step further and approximate fully nonlinear second order PDE (1.1) by the following fourth order quasilinear PDEs [37]¹:

$$(2.1) \quad \varepsilon \Delta^2 u^\varepsilon + F(D^2 u^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x) = 0 \quad \text{in } \Omega, \quad \varepsilon > 0.$$

Here and for the continuation of the paper, we only consider the Dirichlet problem for (1.1), so we suppose that

$$(2.2) \quad u = g \quad \text{on } \partial\Omega.$$

It is then obvious that we need to impose

$$(2.3) \quad u^\varepsilon = g \quad \text{on } \partial\Omega.$$

However, the Dirichlet boundary condition (2.3) is not sufficient for well-posedness, and therefore an additional boundary condition must be used. Several boundary conditions could be used for this purpose, but physically, any additional boundary condition will introduce a so-called “boundary layer”. A better choice would be

¹Other higher order linear operators may be used in the place of $\Delta^2 u^\varepsilon$, we refer the reader to [37] for more discussions on the choices of the regularizing operators. Here, we implicitly assume that $-F$ is elliptic in the sense of [42, Chapter 17], otherwise, (2.1) needs to be replaced by $-\varepsilon \Delta^2 u^\varepsilon + F(D^2 u^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x) = 0$.

one which minimizes the boundary layer. Based on some heuristic arguments and evidence of numerical experiments, we propose to use one of the following additional boundary conditions:

$$(2.4) \quad \Delta u^\varepsilon = \varepsilon \quad \text{on } \partial\Omega,$$

or

$$(2.5) \quad \frac{\partial \Delta u^\varepsilon}{\partial \nu} = \varepsilon \quad \text{on } \partial\Omega,$$

or

$$(2.6) \quad D^2 u^\varepsilon \nu \cdot \nu = \varepsilon \quad \text{on } \partial\Omega,$$

where ν denotes the outward unit normal to $\partial\Omega$.

We note that another valid boundary condition is the following Neumann boundary condition:

$$\frac{\partial u^\varepsilon}{\partial \nu} = \varepsilon \quad \text{on } \partial\Omega.$$

However, since this is an essential boundary condition, it produces a larger boundary layer than the other three boundary conditions, and therefore, we do not recommend the use of this boundary condition.

The rationale for picking boundary condition (2.4) is that we implicitly impose an extra boundary condition

$$\varepsilon^m \Delta u^\varepsilon + u^\varepsilon = g + \varepsilon^{m+1} \quad \text{on } \partial\Omega,$$

which is a higher order perturbation of the original Dirichlet boundary condition (2.2). Intuitively, we expect that the extra boundary condition converges to the original Dirichlet boundary condition as ε tends to zero for sufficiently large positive integer m .

Remark 2.1. (a) We note that boundary conditions (2.4) and (2.5), which are natural boundary conditions for equation (2.1), have an advantage in PDE convergence analysis. Also, both boundary conditions are better suited for conforming and nonconforming finite element methods [39, 62], where as boundary condition in (2.6) fits naturally with the mixed finite element formulation [38].

(b) From the PDE analysis viewpoint, the reason why high order boundary conditions such as (2.4)–(2.6) work better may be explained as follows. Since viscosity solutions generally do not have second or higher order (weak) derivatives, we do not expect u^ε to converge to u in $H^s(\Omega)$ for $s \geq 2$ in general. Therefore, it is possible that errors in higher order derivatives, which could be big, would have small effects on the convergence of u^ε in the lower order norms if u^ε is constructed appropriately. Also, as we shall see later, the reason we do not impose homogeneous boundary conditions in (2.4)–(2.6) is that the regularized solution inherits favorable properties such as strict convexity.

To summarize, the vanishing moment method consists of approximating the (given) nonlinear second order problem

$$(2.7) \quad F(D^2 u, \nabla u, u, x) = 0 \quad \text{in } \Omega,$$

$$(2.8) \quad u = g \quad \text{on } \partial\Omega,$$

by the following quasilinear fourth order boundary value problem:

$$(2.9) \quad \varepsilon \Delta^2 u^\varepsilon + F(D^2 u^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x) = 0 \quad \text{in } \Omega,$$

$$(2.10) \quad u^\varepsilon = g \quad \text{on } \partial\Omega,$$

$$(2.11) \quad \Delta u^\varepsilon = \varepsilon, \quad \text{or} \quad \frac{\partial \Delta u^\varepsilon}{\partial \nu} = \varepsilon, \quad \text{or} \quad D^2 u^\varepsilon \nu \cdot \nu = \varepsilon \quad \text{on } \partial\Omega.$$

Since equation (2.9) is quasilinear, we can then define the notion of a weak solution using the usual integration by parts approach.

Definition 2.2. We define $u^\varepsilon \in H^2(\Omega)$ with $u|_{\partial\Omega} = g$ to be a solution of (2.9)–(2.11)₁ if for all $v \in H^2(\Omega) \cap H_0^1(\Omega)$

$$(2.12) \quad \varepsilon(\Delta u^\varepsilon, \Delta v) + (F(D^2 u^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x), v) = \left\langle \varepsilon^2, \frac{\partial v}{\partial \nu} \right\rangle_{\partial\Omega}.$$

We now are ready to define the notion of moment solutions for (2.7)–(2.8).

Definition 2.3. Suppose that u^ε solves problem (2.9)–(2.11)₁. $\lim_{\varepsilon \rightarrow 0^+} u^\varepsilon$ is called a weak (*resp. strong*) moment solution to problem (2.7)–(2.8) if the convergence holds in H^1 -weak (*resp. H^2 -weak*) topology.

Remark 2.4. (a) The terminologies “moment solutions” and “vanishing moment method” were chosen due to the following consideration. In two-dimensional mechanical applications, u^ε often stands for the vertical displacement of a plate, and $D^2 u^\varepsilon$ is the moment tensor. In the weak formulation, the biharmonic term becomes $\varepsilon(D^2 u^\varepsilon, D^2 v)$ which should vanish as $\varepsilon \rightarrow 0^+$. This is the reason we call $\lim_{\varepsilon \rightarrow 0^+} u^\varepsilon$ (if it exists) a moment solution and call the limiting process the vanishing moment method.

(b) Since weak moment solutions do not have second order weak derivatives in general, they are difficult (if at all possible) to identify. On the other hand, since strong moment solutions do have second order weak derivatives, they are naturally expected to satisfy equation (2.7) almost everywhere and to fulfill the boundary condition (2.10). In the remainder of this book, moment solutions will always mean weak moment solutions.

As problem (2.9)–(2.11) is a quasilinear fourth order problem, one can compute its solutions using literally any well-known numerical methods, in particular, Galerkin-type methods such as finite element methods, spectral and discontinuous Galerkin methods. We note that (2.12) provides a variational formulation for (2.9)–(2.11)₁. Indeed, developing finite element numerical methods is one of two main goals of this book. In Chapter 4 and 5 we shall present comprehensive finite element and mixed finite element analysis for problem (2.7)–(2.8).

However, a natural and larger question is whether the vanishing moment methodology will work. There are two ways to address this question. First, one can do many numerical experiments to see if the methodology works in practice. We indeed have done so (and beyond) in a series of papers [37, 38, 39, 40, 62] (also see [61]) for the Monge-Ampère equation. All numerical experiments of these papers show that the vanishing moment methodology works effectively. Second, one can give a definitive answer to the question by laying down its theoretical foundation, namely, proving the convergence (and rates of convergence if it is possible) (cf. [36]) of the vanishing moment method. Partially accomplishing this goal is in fact the second

main objective of this book. In the next chapter, we shall give a detailed convergence theory for the vanishing moment method applied to the Monge-Ampère equation in the radial symmetric case. We refer the interested reader to [36] for the convergence analysis in more general cases.

We conclude this chapter by mentioning another intriguing property of the vanishing moment method, which was reported in [37] and discovered numerically by accident. When constructing the vanishing moment approximation (2.9), we restrict the parameter ε to be positive (and drive it to zero from the positive side). An interesting question is what happens if we allow ε to be negative (and drive it to zero from the negative side). In other words, we want to know the limiting behaviour as $\varepsilon \searrow 0^+$ of the following problem:

$$(2.13) \quad -\varepsilon \Delta^2 u^\varepsilon + F(D^2 u^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x) = 0 \quad \text{in } \Omega,$$

$$(2.14) \quad u^\varepsilon = g \quad \text{on } \partial\Omega,$$

$$(2.15) \quad \Delta u^\varepsilon = -\varepsilon, \quad \text{or} \quad \frac{\partial \Delta u^\varepsilon}{\partial \nu} = -\varepsilon, \quad \text{or} \quad D^2 u^\varepsilon \nu \cdot \nu = -\varepsilon \quad \text{on } \partial\Omega.$$

The numerical experiments of [37] (also see [61]) indicate that in the case of the two-dimensional Monge-Ampère equation (cf. Chapter 6), that is,

$$F(D^2 v, \nabla v, v, x) = f - \det(D^2 v), \quad f > 0,$$

u^ε converges to the *concave* solution of the Dirichlet problem (2.7)–(2.8)! In the next chapter, we shall also give a proof for this numerical discovery in the radial symmetric case.

CHAPTER 3

Convergence of the vanishing moment method

The primary goal of this chapter is to present a detailed convergence analysis for the vanishing moment method applied to the Monge-Ampère equation in the n -dimensional radial symmetric case. Such a result then puts down the vanishing moment method on a solid footing and provides a (partial) theoretical foundation for the numerical work to be given in the remaining chapters.

3.1. Preliminaries

Unless stated otherwise, throughout this chapter $\Omega = B_R(0) \subset \mathbf{R}^n$ ($n \geq 2$) stands for the ball centered at the origin with radius R . We do not assume Ω is the unit ball because many of our results will depend on the size of the radius R .

Suppose that $f = f(r)$, $f \not\equiv 0$ and $g = g(r)$ in (1.11)–(1.12), that is, f and g are radial. Then the solution u of (1.11)–(1.12) is expected to be radial, namely, $u(x)$ is a function of $r := |x| = \sqrt{\sum_{j=1}^n x_j^2}$. We set $\hat{u}(r) := \hat{u}(|x|) = u(x)$, and for the reader's convenience, we now compute Δu , $\Delta^2 u$ and $\det(D^2 u)$ in terms of \hat{u} (cf. [59, 70]). Trivially,

$$\frac{\partial r}{\partial x_j} = \frac{x_j}{r}, \quad \frac{\partial}{\partial x_j} \left(\frac{1}{r} \right) = -\frac{x_j}{r^3}.$$

By the chain rule we have

$$\begin{aligned} \frac{\partial u(x)}{\partial x_j} &= \hat{u}_r(r) \frac{\partial r}{\partial x_j} = \hat{u}_r(r) \frac{x_j}{r}, \\ \frac{\partial^2 u(x)}{\partial x_j \partial x_i} &= \frac{x_j}{r} \frac{\partial}{\partial x_i} \hat{u}_r(r) + \hat{u}_r(r) \frac{\partial}{\partial x_i} \left(\frac{x_j}{r} \right) = \frac{1}{r} \left(\frac{1}{r} \hat{u}_r(r) \right)_r x_i x_j + \frac{\hat{u}_r(r)}{r} \delta_{ij}. \end{aligned}$$

Here, the subscripts stand for the derivatives with respect to the subscript variables.

On noting that $D^2 u(x)$ is a diagonal perturbation of a scaled rank-one matrix xx^T , and since the eigenvalues of xx^T are 0 (with multiplicity $n - 1$) and $|x|^2 = r^2$ (with multiplicity 1 and corresponding eigenvector x), then the eigenvalues of $D^2 u(x)$ are

$$\begin{aligned} \lambda_1 &:= \frac{\hat{u}_r(r)}{r} + r \left(\frac{1}{r} \hat{u}_r(r) \right)_r = \hat{u}_{rr}(r) \quad (\text{with multiplicity } 1), \\ \lambda_2 &:= \frac{\hat{u}_r(r)}{r} \quad (\text{with multiplicity } n - 1). \end{aligned}$$

Thus,

$$\begin{aligned}\Delta u(x) &= \lambda_1 + (n-1)\lambda_2 = \hat{u}_{rr}(r) + \frac{n-1}{r}\hat{u}_r(r) = \frac{1}{r^{n-1}}(r^{n-1}\hat{u}_r)_r, \\ \Delta^2 u(x) &= \Delta(\Delta u) = \Delta\left(\frac{1}{r^{n-1}}(r^{n-1}\hat{u}_r)_r\right) = \frac{1}{r^{n-1}}\left(r^{n-1}\left(\frac{1}{r^{n-1}}(r^{n-1}\hat{u}_r)_r\right)_r\right)_r, \\ \det(D^2 u(x)) &= \lambda_1(\lambda_2)^{n-1} = \hat{u}_{rr}(r)\left[\frac{\hat{u}_r(r)}{r}\right]^{n-1} = \frac{1}{nr^{n-1}}((\hat{u}_r)^n)_r.\end{aligned}$$

Abusing the notion to denote $\hat{u}(r)$ by $u(r)$, then problem (1.11)–(1.12) becomes seeking a function $u = u(r)$ such that

$$(3.1) \quad \frac{1}{nr^{n-1}}((u_r)^n)_r = f \quad \text{in } (0, R),$$

$$(3.2) \quad u(R) = g(R),$$

$$(3.3) \quad u_r(0) = 0.$$

We remark that boundary condition (3.3) is due to the symmetry of $u = u(r)$.

Lemma 3.1. *Suppose that $r^{n-1}f \in L^1((0, R))$ and $f \geq 0$ a.e. on $(0, R)$. Then there exists exactly one real solution if n is odd and there are exactly two real solutions if n is even, to the boundary value problem (3.1)–(3.3). Moreover, the solutions are given by the formula*

$$(3.4) \quad u(r) = \begin{cases} g(R) \pm \int_r^R (nL_f(s))^{\frac{1}{n}} ds & \text{if } n \text{ is even,} \\ g(R) - \int_r^R (nL_f(s))^{\frac{1}{n}} ds & \text{if } n \text{ is odd} \end{cases}$$

for $r \in (0, R)$. Where

$$(3.5) \quad L_f(s) := \int_0^s t^{n-1} f(t) dt.$$

Since the proof is elementary (cf. [59, 70]), we omit it. Clearly, when n is even, the first solution (with “+” sign) is concave and the second solution (with “−” sign) is convex because u_r and u_{rr} simultaneously positive and negative respectively in the two cases. When n is odd, the real solution is convex.

Remark 3.2. The above theorem shows that u is C^2 at a point $r_0 \in (0, R)$ as long as f is C^0 at r_0 and $L_f(r_0) \neq 0$. Also, u is smooth in $(0, R)$ if f is smooth in $(0, R)$. We refer the reader to [59, 70] for the precise conditions on f at $r = 0$ to ensure the regularity of u at $r = 0$, extensions to the complex Monge-Ampère equation, and generalized Monge-Ampère equations in which $f = f(\nabla u, u, x)$.

Similarly, it is expected that $u^\varepsilon = u^\varepsilon(r)$ is also radial, and the vanishing moment approximation (2.9)–(2.11)₁ then becomes (cf. Chapter 6)

$$(3.6) \quad -\frac{\varepsilon}{r^{n-1}}\left(\frac{1}{r^{n-1}}(r^{n-1}u_r^\varepsilon)_r\right)_r + \frac{1}{nr^{n-1}}((u_r^\varepsilon)^n)_r = f \quad \text{in } (0, R),$$

$$(3.7) \quad u^\varepsilon(R) = g(R),$$

$$(3.8) \quad u_r^\varepsilon(0) = 0, \quad |u_{rr}^\varepsilon(0)| < \infty, \quad |u_{rrr}^\varepsilon(r)| = o\left(\frac{1}{r^{n-1}}\right) \quad \text{as } r \rightarrow 0^+,$$

$$(3.9) \quad u_{rr}^\varepsilon(R) + \frac{n-1}{R}u_r^\varepsilon(R) = \varepsilon.$$

Later in this chapter, we shall analyze problem (3.6)–(3.9) which includes proving its existence and uniqueness as well as regularities. After this is done, we then show that the solution u^ε of (3.6)–(3.9) converges to the unique convex solution of (3.1)–(3.3).

Integrating over $(0, r)$ after multiplying (3.6) by r^{n-1} , using boundary condition (3.8) and

$$\lim_{r \rightarrow 0^+} r^{n-1} \left(u_{rrr}^\varepsilon + \frac{n-1}{r} u_{rr}^\varepsilon - \frac{n-1}{r^2} u_r^\varepsilon \right) = 0$$

we get

$$(3.10) \quad -\varepsilon r^{n-1} \left(\frac{1}{r^{n-1}} (r^{n-1} u_r)_r \right)_r + \frac{1}{n} (u_r^\varepsilon)^n = L_f \quad \text{in } (0, R).$$

Introduce the new function $w^\varepsilon(r) := r^{n-1} u_r^\varepsilon(r)$. A direct calculation shows that w^ε satisfies

$$(3.11) \quad -\varepsilon r^{n-1} \left(\frac{1}{r^{n-1}} w_r^\varepsilon \right)_r + \frac{1}{nr^{n(n-1)}} (w^\varepsilon)^n = L_f \quad \text{in } (0, R).$$

Converting the boundary conditions (3.8)–(3.9) to w^ε we have

$$(3.12) \quad w^\varepsilon(0) = w_r^\varepsilon(0) = 0,$$

$$(3.13) \quad w_r^\varepsilon(R) = \varepsilon R^{n-1}.$$

In addition, since

$$\begin{aligned} w_r^\varepsilon &= r^{n-1} u_{rr}^\varepsilon + (n-1)r^{n-2} u_r^\varepsilon, \\ w_{rr}^\varepsilon &= r^{n-1} u_{rrr}^\varepsilon + 2(n-1)r^{n-2} u_{rr}^\varepsilon + (n-1)(n-2)r^{n-3} u_r^\varepsilon, \end{aligned}$$

we have

$$(3.14) \quad \frac{\partial^j w^\varepsilon}{\partial r^j}(0) = o\left(\frac{1}{r^{n-1-j}}\right) \quad \text{for } 0 \leq j \leq \min\{2, n-1\}.$$

So we have derived from (3.6) a reduced equation (3.11), which is only of second order, hence, it is easier to handle. After problem (3.11)–(3.13) is fully understood, we then come back to analyze problem (3.6)–(3.9).

3.2. Existence, uniqueness, and regularity of vanishing moment approximations

We now prove that problem (3.11)–(3.13) possesses a unique nonnegative classical solution. First, we state and prove the following uniqueness result.

Theorem 3.3. *Problem (3.11)–(3.13) has at most one nonnegative classical solution.*

PROOF. Suppose that w_1^ε and w_2^ε are two nonnegative classical solutions to (3.11)–(3.13). Let

$$\phi^\varepsilon := w_1^\varepsilon - w_2^\varepsilon \quad \text{and} \quad \bar{w}^\varepsilon := \begin{cases} \sum_{\substack{\alpha+\beta=n-1 \\ \alpha, \beta \geq 0}} (w_1^\varepsilon)^\alpha (w_2^\varepsilon)^\beta & \text{if } w_1^\varepsilon = w_2^\varepsilon \\ \frac{(w_1^\varepsilon)^n - (w_2^\varepsilon)^n}{w_1^\varepsilon - w_2^\varepsilon} & \text{otherwise.} \end{cases}$$

Subtracting the corresponding equations satisfied by w_1^ε and w_2^ε yields

$$(3.15) \quad -\varepsilon r^{n-1} \left(\frac{1}{r^{n-1}} \phi_r^\varepsilon \right)_r + \frac{1}{nr^{n(n-1)}} \bar{w}^\varepsilon \phi^\varepsilon = 0 \quad \text{in } (0, R),$$

$$(3.16) \quad \phi^\varepsilon(0) = \phi_r^\varepsilon(0) = 0,$$

$$(3.17) \quad \phi_r^\varepsilon(R) = 0.$$

On noting that $\bar{w}^\varepsilon \geq 0$ in $[0, R]$, by the weak maximum principle [32, Theorem 2, page 329] we conclude

$$\max_{[0, R]} |\phi^\varepsilon(r)| = \max\{|\phi^\varepsilon(0)|, |\phi^\varepsilon(R)|\} = \max\{0, |\phi^\varepsilon(R)|\}.$$

If $\phi^\varepsilon(R) = 0$, then $\phi^\varepsilon \equiv 0$. If $\phi^\varepsilon(R) \neq 0$, then ϕ^ε takes its maximum or minimum value at $r = R$. However, the strong maximum principle [69, Theorem 4, page 7] implies that $\phi_r^\varepsilon(R) \neq 0$, which contradicts with boundary condition $\phi_r^\varepsilon(R) = 0$. Hence, $\phi^\varepsilon \equiv 0$ or $w_1^\varepsilon \equiv w_2^\varepsilon$. The proof is complete. \square

Remark 3.4. A more direct way to prove $\phi^\varepsilon \equiv 0$ is given as follows. Multiplying (3.15) by ϕ^ε , integrating by parts, and using the boundary conditions (3.16)–(3.17) yield

$$(3.18) \quad \varepsilon \int_0^R |\phi_r^\varepsilon(r)|^2 dr + \varepsilon \int_0^R \frac{n-1}{r} \phi_r^\varepsilon(r) \phi^\varepsilon(r) dr \\ + \int_0^R \frac{1}{nr^{n(n-1)}} \bar{w}^\varepsilon(r) |\phi^\varepsilon(r)|^2 dr = 0.$$

On noting that

$$\int_0^R \frac{n-1}{r} \phi_r^\varepsilon(r) \phi^\varepsilon(r) dr = \frac{n-1}{2r} (\phi^\varepsilon(r))^2 \Big|_{r=0}^{r=R} + \int_0^R \frac{n-1}{2r^2} (\phi^\varepsilon(r))^2 dr \\ = \frac{n-1}{2R} (\phi^\varepsilon(R))^2 + \int_0^R \frac{n-1}{2r^2} (\phi^\varepsilon(r))^2 dr.$$

so each term on the left-hand side of (3.18) is nonnegative, hence, they all must be zero. The first term then gives $\phi_r^\varepsilon \equiv 0$. Then $\phi^\varepsilon \equiv \text{const}$. Hence, $\phi^\varepsilon \equiv 0$ by $\phi^\varepsilon(0) = 0$.

Next, we prove that the existence of nonnegative solutions to problem (3.11)–(3.13).

Theorem 3.5. *Suppose $r^{n-1}f \in L^1((0, R))$ and $f \geq 0$ a.e. in $(0, R)$, then there is a nonnegative classical solution to problem (3.11)–(3.13).*

PROOF. We divide the proof into three steps.

Step 1: Let $\psi^0 \in C^2([0, R])$ be nonnegative and satisfy $\psi^0(0) = \psi_r^0(0) = 0$ and $\psi_r^0(R) = \varepsilon R^{n-1}$. One such an example is $\psi^0(r) = \frac{\varepsilon}{n} r^n$. We then define a sequence of functions $\{\psi^k\}_{k \geq 0}$ recursively by solving for $k = 0, 1, 2, \dots$

$$(3.19) \quad -\varepsilon r^{n-1} \left(\frac{1}{r^{n-1}} \psi_r^{k+1} \right)_r + \frac{1}{nr^{n(n-1)}} (\psi^k)^{n-1} \psi^{k+1} \\ = L_f(r) := \int_0^r s^{n-1} f(s) ds \quad \text{in } (0, R),$$

$$(3.20) \quad \psi^{k+1}(0) = \psi_r^{k+1}(0) = 0,$$

$$(3.21) \quad \psi_r^{k+1}(R) = \varepsilon R^{n-1}.$$

We first show by induction that for any such sequence satisfying (3.19)–(3.21), there holds $\psi^k \geq 0$ in $[0, R]$ for all $k \geq 0$. Note that $\psi^0 \geq 0$ by construction. Suppose that $\psi^k \geq 0$ in $[0, R]$. Since $f \geq 0$ and $f \not\equiv 0$, then

$$-\varepsilon r^{n-1} \left(\frac{1}{r^{n-1}} \psi_r^{k+1} \right)_r + \frac{1}{nr^{n(n-1)}} (\psi^k)^{n-1} \psi^{k+1} > 0 \quad \text{in } (0, R).$$

Hence, ψ^{k+1} is a supersolution to the linear differential operator on the left-hand side of (3.19). By the weak maximum principle [32, Theorem 2, page 329] we have

$$\min_{[0, R]} \psi^{k+1}(r) \geq \min\{0, \psi^{k+1}(0), \psi^{k+1}(R)\} = \min\{0, \psi^{k+1}(R)\}.$$

If $\psi^{k+1}(R) < 0$, and since $\psi_r^{k+1}(R) = \varepsilon R^{n-1} > 0$, the strong maximum principle [69, Theorem 4, page 7] implies that $\psi^{k+1} \equiv \text{const}$ in $[0, R]$, which leads to a contradiction as $\psi^{k+1}(0) = 0$. Thus, we must have $\psi^{k+1}(R) \geq 0$, and therefore, $\psi^{k+1} \geq 0$ in $[0, R]$. By the induction argument, we conclude that $\psi^k \geq 0$ in $[0, R]$ for all $k \geq 0$.

It then follows from the standard theory for linear elliptic equations (cf. [32, 42]) that (3.19)–(3.21) has a unique classical solution ψ^{k+1} . Hence the $(k+1)$ th iterate ψ^{k+1} is well defined, and therefore, so is the sequence $\{\psi^k\}_{k \geq 0}$.

Step 2: Next, we shall derive some uniform (in k) estimates for the sequence $\{\psi^k\}_{k \geq 0}$. To this end, we first prove that $\psi^{k+1}(R)$ can be bounded from above uniformly in k . Multiplying (3.19) by ψ^{k+1} and integrating by parts yield

$$\begin{aligned} (3.22) \quad & -\varepsilon \psi_r^{k+1}(r) \psi^{k+1}(r) \Big|_{r=0}^{r=R} + \varepsilon \int_0^R |\psi_r^{k+1}(r)|^2 dr \\ & + \varepsilon \int_0^R \frac{n-1}{r} \psi_r^{k+1}(r) \psi^{k+1}(r) dr + \int_0^R \frac{1}{nr^{n(n-1)}} (\psi^k(r))^{n-1} |\psi^{k+1}(r)|^2 dr \\ & = \int_0^R \psi^{k+1}(r) L_f(r) dr. \end{aligned}$$

It follows from boundary conditions (3.20) and (3.21) that

$$(3.23) \quad -\varepsilon \psi_r^{k+1}(r) \psi^{k+1}(r) \Big|_{r=0}^{r=R} = -\varepsilon^2 R^{n-1} \psi^{k+1}(R).$$

Integrating by parts gives

$$\begin{aligned} (3.24) \quad & \int_0^R \frac{1}{r} \psi_r^{k+1}(r) \psi^{k+1}(r) dr = \frac{1}{2r} (\psi^{k+1}(r))^2 \Big|_{r=0}^{r=R} + \int_0^R \frac{1}{2r^2} (\psi^{k+1}(r))^2 dr \\ & = \frac{1}{2R} (\psi^{k+1}(R))^2 + \int_0^R \frac{1}{2r^2} (\psi^{k+1}(r))^2 dr. \end{aligned}$$

By Schwarz, Poincaré, and Young's inequalities, we get

$$(3.25) \quad \int_0^R \psi^{k+1}(r) L_f(r) dr \leq \frac{\varepsilon}{2} \int_0^R |\psi_r^{k+1}(r)|^2 dr + \frac{C_1^2}{2\varepsilon} \int_0^R (L_f(r))^2 dr$$

for some positive constant $C_1 = C_1(R)$.

Combining (3.22)–(3.25) we obtain

$$\begin{aligned}
 (3.26) \quad & -2\varepsilon^2 R^{n-1} \psi^{k+1}(R) + \varepsilon \int_0^R |\psi_r^{k+1}(r)|^2 dr + \frac{\varepsilon(n-1)}{R} (\psi^{k+1}(R))^2 \\
 & + \int_0^R \frac{\varepsilon(n-1)}{r^2} (\psi^{k+1}(r))^2 dr + \int_0^R \frac{2}{nr^{n(n-1)}} (\psi^k(r))^{n-1} |\psi^{k+1}(r)|^2 dr \\
 & \leq \frac{C_1^2}{\varepsilon} \int_0^R (L_f(r))^2 dr.
 \end{aligned}$$

Let

$$z := \psi^{k+1}(R), \quad b := \frac{2\varepsilon R^n}{n-1}, \quad c := \frac{C_1^2 R^2}{\varepsilon^2(n-1)} (L_f(R))^2.$$

Then from (3.26) we have

$$z^2 - bz - c \leq 0,$$

which in turn implies that

$$z_1 \leq z \leq z_2, \quad \text{where} \quad z_1 = \frac{b - \sqrt{b^2 + 4c}}{2}, \quad z_2 = \frac{b + \sqrt{b^2 + 4c}}{2}.$$

Since $-z_2 < z_1$, the above inequality then infers that $|z| \leq z_2$. Thus, there exists a positive constant $C_2 = C_2(R, L_f)$ such that

$$(3.27) \quad |\psi^{k+1}(R)| \leq z_2 \leq \frac{C_2}{\varepsilon}.$$

Substituting (3.27) into the first term on the left-hand side of (3.26) we also get

$$\begin{aligned}
 (3.28) \quad & \varepsilon \int_0^R |\psi_r^{k+1}(r)|^2 dr + \int_0^R \frac{\varepsilon(n-1)}{r^2} (\psi^{k+1}(r))^2 dr \\
 & + \int_0^R \frac{2}{nr^{n(n-1)}} (\psi^k(r))^{n-1} |\psi^{k+1}(r)|^2 dr \\
 & \leq C_3 := \frac{C_1^2 R}{\varepsilon} (L_f(R))^2 + 2\varepsilon^2 R^{n-1} C_2.
 \end{aligned}$$

Now using the pointwise estimate for linear elliptic equations [42, Theorem 3.7] we have

$$(3.29) \quad \max_{[0, R]} |\psi^{k+1}(r)| \leq \left(\frac{C_2}{\varepsilon} + \frac{L_f(R)}{\varepsilon} \right).$$

Next, we show that ψ_r^{k+1} is also uniformly bounded (in k) in $[0, R]$. To this end, integrating (3.19) over $(0, r)$ after multiplying it by $r^{n(n-1)}$, and integrating by parts twice in the first term yield

$$\begin{aligned}
 (3.30) \quad \psi_r^{k+1}(r) = & -\frac{(n^2-1)[n(n-1)-1]}{r^{n(n-1)}} \int_0^r s^{n(n-1)-2} \psi^{k+1}(s) ds \\
 & + \frac{n^2-1}{r} \psi^{k+1}(r) + \frac{1}{\varepsilon nr^{n(n-1)}} \int_0^r (\psi^k(s))^{n-1} \psi^{k+1}(s) ds \\
 & - \frac{1}{\varepsilon r^{n(n-1)}} \int_0^r s^{n(n-1)} L_f(s) ds \quad \forall r \in (0, R).
 \end{aligned}$$

Using L'Hôpital's rule it is easy to check that the limit as $r \rightarrow 0^+$ of each term on the right-hand side of (3.30) is zero, hence, each term is bounded in a neighborhood of $r = 0$. Moreover, on noting that $\psi^k \geq 0$, by Schwarz inequality, we have

$$(3.31) \quad \int_0^r (\psi^k(s))^{n-1} \psi^{k+1}(s) ds \\ \leq \left(\int_0^r s^{n(n-1)} (\psi^k(s))^{n-1} ds \right)^{\frac{1}{2}} \left(\int_0^r \frac{1}{s^{n(n-1)}} (\psi^k(s))^{n-1} |\psi^{k+1}(s)|^2 ds \right)^{\frac{1}{2}}.$$

Now in view of (3.28)–(3.31) we conclude that there exists a positive constant $C_4 = C_4(R, L_f)$ such that

$$(3.32) \quad \max_{[0, R]} |\psi_r^{k+1}(r)| \leq \frac{C_4}{\varepsilon^{\frac{n+2}{2}}}.$$

By (3.19) we get

$$(3.33) \quad \psi_{rr}^{k+1}(r) = \frac{1}{r} \psi_r^{k+1}(r) + \frac{1}{\varepsilon n r^{n(n-1)}} (\psi^k(r))^{n-1} \psi^{k+1}(r) \\ - \frac{1}{\varepsilon} L_f(r) \quad \forall r \in (0, R).$$

Again, using L'Hôpital's rule and (3.14) it is easy to check that the limit as $r \rightarrow 0^+$ of each term on the right-hand side of (3.33) exists, and therefore, each term is bounded in a neighborhood of $r = 0$. Hence, it follows from (3.29) and (3.32) that there exists a positive constant $C_5 = C_5(R, L_f)$ such that

$$(3.34) \quad \max_{[0, R]} |\psi_{rr}^{k+1}(r)| \leq \frac{C_5}{\varepsilon^{n+1}}.$$

To summarize, we have proved that $\|\psi^{k+1}\|_{C^j([0, R])} \leq C(\varepsilon, R, n, L_f)$ for $j = 0, 1, 2$ and the bounds are independent of k . Clearly, by a simple induction argument we conclude that these estimates hold for all $k \geq 0$.

Step 3: Since $\|\psi^k\|_{C^2([0, R])}$ is uniformly bounded in k , then both $\{\psi^k\}_{k \geq 0}$ and $\{\psi_r^k\}_{k \geq 0}$ are uniformly equicontinuous. It follows from Arzela-Ascoli compactness theorem (cf. [32, page 635]) that there is a subsequence of $\{\psi^k\}_{k \geq 0}$ (still denoted by the same notation) and $\psi \in C^2([0, R])$ such that

$$\begin{aligned} \psi^k &\longrightarrow \psi && \text{uniformly in every compact set } E \subset (0, R) \text{ as } k \rightarrow \infty, \\ \psi_r^k &\longrightarrow \psi_r && \text{uniformly in every compact set } E \subset (0, R) \text{ as } k \rightarrow \infty. \end{aligned}$$

Testing equation (3.19) with an arbitrary function $\chi \in C_0^1((0, R))$ yields

$$\begin{aligned} \varepsilon \int_0^R \psi_r^{k+1}(r) \chi_r(r) dr + \varepsilon \int_0^R \frac{n-1}{r} \psi_r^{k+1}(r) \chi(r) dr \\ + \int_0^R \frac{1}{n r^{n(n-1)}} (\psi^k(r))^{n-1} \psi^{k+1}(r) \chi(r) dr = \int_0^R L_f(r) \chi(r) dr. \end{aligned}$$

Setting $k \rightarrow \infty$ and using the Lebesgue Dominated Convergence Theorem, we get

$$(3.35) \quad \varepsilon \int_0^R \psi_r(r) \chi_r(r) dr + \varepsilon \int_0^R \frac{n-1}{r} \psi_r(r) \chi(r) dr \\ + \int_0^R \frac{1}{n r^{n(n-1)}} (\psi(r))^n \chi(r) dr = \int_0^R L_f(r) \chi(r) dr.$$

Since $\psi \in C^2([0, R])$, we are able to integrate by parts in the first term on the left-hand side of (3.35), yielding

$$\int_0^R \left[-\varepsilon \psi_{rr}(r) + \frac{\varepsilon(n-1)}{r} \psi_r(r) + \frac{1}{nr^{n(n-1)}} (\psi(r))^n - L_f(r) \right] \chi(r) dr = 0$$

for all $\chi \in C_0^1((0, R))$. This then implies that

$$-\varepsilon \psi_{rr}(r) + \frac{\varepsilon(n-1)}{r} \psi_r(r) + \frac{1}{nr^{n(n-1)}} (\psi(r))^n - L_f(r) = 0 \quad \forall r \in (0, R),$$

that is,

$$-\varepsilon r^{n-1} \left(\frac{1}{r^{n-1}} \psi_r(r) \right)_r + \frac{1}{nr^{n(n-1)}} (\psi(r))^n = L_f(r) \quad \forall r \in (0, R).$$

Thus, ψ satisfies (3.11) pointwise in $(0, R)$.

Finally, it is clear that $\psi \geq 0$ in $[0, R]$, and it follows easily from (3.20) and (3.21) that

$$\psi(0) = \psi_r(0) = 0 \quad \text{and} \quad \psi_r(R) = \varepsilon R^{n-1}.$$

So we have demonstrated that $\psi \in C^2([0, R])$ is a nonnegative classical solution to problem (3.11)–(3.13). The proof is complete. \square

Remark 3.6. (a) The proof at the beginning of *Step 2* gives an estimate for the Neumann to Dirichlet map: $\psi_r^{k+1}(R) \rightarrow \psi^{k+1}(R)$.

(b) We note that the a priori estimates derived in the proof are not sharp in ε . Better estimates will be obtained (and needed) in the next section after the positivity of Δu^ε is established.

The above proof together with the uniqueness theorem, Theorem 3.3, and the strong maximum principle immediately give the following corollary.

Corollary 3.7. *Suppose $r^{n-1}f \in L^1((0, R))$ and $f \geq 0$ a.e. in $(0, R)$, then there exists a unique nonnegative classical solution w^ε to problem (3.11)–(3.13). Moreover, $w^\varepsilon > 0$ in $(0, R)$, $w^\varepsilon \in C^3((0, R))$ if $f \in C^0((0, R))$, and w^ε is C^∞ provided that f is C^∞ .*

Recall that $w^\varepsilon = r^{n-1}u_r^\varepsilon$ where u^ε and w^ε are solutions of (3.6)–(3.9) and (3.11)–(3.13). Let w^ε be the unique solution to (3.11)–(3.13), as stated in Corollary 3.7, define

$$(3.36) \quad u^\varepsilon(r) := g(R) - \int_r^R \frac{1}{s^{n-1}} w^\varepsilon(s) ds \quad \forall r \in (0, R).$$

We now show that u^ε is a unique monotone increasing classical solution of problem (3.6)–(3.9).

Theorem 3.8. *Suppose $f \in C^0((0, R))$ and $f \geq 0$ in $(0, R)$, then problem (3.6)–(3.9) has a unique monotone increasing classical solution. Moreover, u^ε is smooth provided that f is smooth.*

PROOF. By direct calculations one can easily show that u^ε defined by (3.36) satisfies (3.6)–(3.9). Since $u_r^\varepsilon > 0$ in $(0, R)$, then u^ε is a monotone increasing function. Hence, the existence is done.

To show uniqueness, we notice that u^ε is a monotone increasing classical solution of problem (3.6)–(3.9) if and only if w^ε is a nonnegative classical solution of problem (3.11)–(3.13). Hence, the uniqueness of (3.6)–(3.9) follows from the uniqueness of (3.11)–(3.13). The proof is complete. \square

3.3. Convexity of vanishing moment approximations

The goal of this section is to analyze the convexity of the solution u^ε whose existence is proved in Theorem 3.8. We shall prove that u^ε is strictly convex either in $(0, R)$ or in $(0, R - c_0\varepsilon)$ for some ε -independent positive constant c_0 . From calculations of Section 3.1 we know that D^2u^ε only has two distinct eigenvalues $\lambda_1 = u_{rr}^\varepsilon$ (with multiplicity 1) and $\lambda_2 = \frac{1}{r}u_r^\varepsilon$ (with multiplicity $n-1$), and we have proved that $\lambda_2 \geq 0$ in $(0, R)$, so it is necessary to show $\lambda_1 \geq 0$ in $(0, R)$ or in $(0, R - c_0\varepsilon)$. In addition, in this section we derive some sharp uniform (in ε) a priori estimates for the vanishing moment approximations u^ε , which will play an important role not only for establishing the convexity property for u^ε but also for proving the convergence of u^ε in the next section.

First, we have the following positivity result for Δu^ε .

Theorem 3.9. *Let u^ε be the unique monotone increasing classical solution of problem (3.6)–(3.9) and define $w^\varepsilon := r^{n-1}u_r^\varepsilon$. Then*

- (i) $w_r^\varepsilon > 0$ in $(0, R)$, consequently, $\Delta u^\varepsilon > 0$ in $(0, R)$, for all $\varepsilon > 0$.
- (ii) For any $r_0 \in (0, R)$, there exists an $\varepsilon_0 > 0$ such that $w_r^\varepsilon > \varepsilon r^{n-1}$ and $\Delta u^\varepsilon > \varepsilon$ in (r_0, R) for $\varepsilon \in (0, \varepsilon_0)$.

PROOF. We split the proof into two steps.

Step 1: Since u^ε is monotone increasing and differentiable, then $u_r^\varepsilon \geq 0$ in $[0, R]$. From the derivation of Section 3.1 we know that $w^\varepsilon := r^{n-1}u_r^\varepsilon$ is the unique nonnegative classical solution of (3.11)–(3.13). Let $\varphi^\varepsilon := w_r^\varepsilon$. By the definition of the Laplacian Δ we have

$$(3.37) \quad \varphi^\varepsilon = w_r^\varepsilon = r^{n-1}u_{rr}^\varepsilon + (n-1)r^{n-2}u_r^\varepsilon = r^{n-1}\Delta u^\varepsilon.$$

So $\varphi^\varepsilon > 0$ in $(0, R)$ infers $\Delta u^\varepsilon > 0$ in $(0, R)$.

To show $\varphi^\varepsilon > 0$, we differentiate (3.11) with respect to r to get

$$\begin{aligned} -\varepsilon w_{rrr}^\varepsilon + \varepsilon(n-1)r^{n-2}\left(\frac{1}{r^{n-1}}w_r^\varepsilon\right)_r + \left[\frac{\varepsilon(n-1)(n-2)}{r^2} + \frac{(w^\varepsilon)^{n-1}}{r^{n(n-1)}}\right]w_r^\varepsilon \\ - \frac{n-1}{r^{(n-1)^2+n}}(w^\varepsilon)^n = r^{n-1}f(r) \quad \text{in } (0, R). \end{aligned}$$

From (3.11), we have

$$\varepsilon r^{n-2}\left(\frac{1}{r^{n-1}}w_r^\varepsilon\right)_r = \frac{1}{nr^{(n-1)^2+n}}(w^\varepsilon)^n - \frac{1}{r}L_f.$$

Combining the above two equations yields

$$\begin{aligned} (3.38) \quad -\varepsilon w_{rrr}^\varepsilon + \left[\frac{\varepsilon(n-1)(n-2)}{r^2} + \frac{(w^\varepsilon)^{n-1}}{r^{n(n-1)}}\right]w_r^\varepsilon \\ = r^{n-1}f + \frac{n-1}{r}L_f + \frac{(n-1)^2}{nr^{(n-1)^2+n}}(w^\varepsilon)^n. \end{aligned}$$

Substituting $w_r^\varepsilon = \varphi^\varepsilon$ into the above equation we get

$$\begin{aligned} (3.39) \quad -\varepsilon \varphi_{rrr}^\varepsilon + \left[\frac{\varepsilon(n-1)(n-2)}{r^2} + \frac{(w^\varepsilon)^{n-1}}{r^{n(n-1)}}\right]\varphi^\varepsilon \\ = r^{n-1}f + \frac{n-1}{r}L_f + \frac{(n-1)^2}{nr^{(n-1)^2+n}}(w^\varepsilon)^n \geq 0 \quad \text{in } (0, R), \end{aligned}$$

since $f, L_f, w^\varepsilon \geq 0$ in $(0, R)$. This means that φ^ε is a supersolution to a linear uniformly elliptic differential operator. By the weak maximum principle we get (cf. [32, page 329])

$$\min_{[0, R]} \varphi^\varepsilon(r) \geq \min\{0, \varphi^\varepsilon(0), \varphi^\varepsilon(R)\} = \min\{0, 0, R^{n-1}\varepsilon\} = 0.$$

Here we have used the fact that $\varphi^\varepsilon(R) = R^{n-1}\Delta u^\varepsilon(R) = R^{n-1}\varepsilon$. Hence, $\varphi^\varepsilon \geq 0$ in $[0, R]$, so $\Delta u^\varepsilon \geq 0$ in $[0, R]$.

It follows from the strong maximum principle (cf. [69, Theorem 4, page 7]) that φ^ε can not attain its nonpositive minimum value 0 at any point in $(0, R)$. Therefore, $\varphi^\varepsilon > 0$ in $(0, R)$, which implies that $\Delta u^\varepsilon > 0$ in $(0, R)$. So assertion (i) holds.

Step 2: To show (ii), let $\psi^\varepsilon := w_r^\varepsilon - \varepsilon r^{n-1} = r^{n-1}(\Delta u^\varepsilon - \varepsilon)$. Using the identities

$$w_r^\varepsilon = \psi^\varepsilon + \varepsilon r^{n-1}, \quad w_{rrr}^\varepsilon = \psi_{rr}^\varepsilon + \varepsilon(n-1)(n-2)r^{n-3},$$

we rewrite (3.38) as

$$(3.40) \quad -\varepsilon\psi_{rr}^\varepsilon + \left[\frac{\varepsilon(n-1)(n-2)}{r^2} + \frac{(w^\varepsilon)^{n-1}}{r^{n(n-1)}} \right] \psi^\varepsilon \\ = r^{n-1}f + \frac{n-1}{r}L_f + \frac{(w^\varepsilon)^{n-1}[(n-1)^2w^\varepsilon - \varepsilon nr^n]}{nr^{(n-1)^2+n}} \quad \text{in } (0, R).$$

Hence, ψ^ε satisfies a linear uniformly elliptic equation.

Now, on noting that $w^\varepsilon \geq 0$ by (i), for any $r_0 \in (0, R)$ (i.e., r_0 is away from 0), it is easy to see that there exists an $\varepsilon_1 > 0$ such that the right-hand side of (3.40) is nonnegative in (r_0, R) for all $\varepsilon \in (0, \varepsilon_1)$. Hence, ψ^ε is a supersolution in (r_0, R) to the uniformly elliptic operator on the right-hand side of (3.40). By the weak maximum principle we have (cf. [32, page 329])

$$\min_{[r_0, R]} \psi^\varepsilon(r) \geq \min\{0, \psi^\varepsilon(r_0), \psi^\varepsilon(R)\} = \min\{0, \Delta u^\varepsilon(r_0) - \varepsilon, 0\}.$$

Again, here we have used the fact that $\Delta u^\varepsilon(R) = \varepsilon$.

Since $\Delta u^\varepsilon(r_0) > 0$, choose $\varepsilon_0 = \min\{\varepsilon_1, \frac{1}{2}\Delta u^\varepsilon(r_0)\}$, then $\psi^\varepsilon(r_0) = \Delta u^\varepsilon(r_0) - \varepsilon \geq \frac{1}{2}\Delta u^\varepsilon(r_0) > 0$ for $\varepsilon \in (0, \varepsilon_0)$. Thus, $\min_{[r_0, R]} \psi^\varepsilon(r) \geq 0$ for $\varepsilon \in (0, \varepsilon_0)$. Therefore, $w_r^\varepsilon \geq \varepsilon r^{n-1}$, consequently, $\Delta u^\varepsilon \geq \varepsilon$ in $[r_0, R]$ for $\varepsilon \in (0, \varepsilon_0)$.

Finally, an application of the strong maximum principle (cf. [69, Theorem 4, page 7]) yields that $w_r^\varepsilon > \varepsilon r^{n-1}$, hence $\Delta u^\varepsilon > \varepsilon$, in (r_0, R) for $\varepsilon \in (0, \varepsilon_0)$. The proof is complete. \square

Remark 3.10. The proof also shows that ε_0 decreases (resp. increases) as r_0 decreases (resp. increases), and $v^\varepsilon := \Delta u^\varepsilon$ takes its minimum value ε in $[r_0, R]$ at the right end of the interval $r = R$.

With help of the positivity of Δu^ε , we can derive some better uniform estimates (in ε) for w^ε and u^ε .

Theorem 3.11. *Suppose $f \in C^0((0, R))$ and $f \geq 0$ in $(0, R)$. Let u^ε be the unique monotone increasing classical solution to problem (3.6)–(3.9). Define $w^\varepsilon = r^{n-1}u_r^\varepsilon$ and $v^\varepsilon = \Delta u^\varepsilon = u_{rr}^\varepsilon + \frac{n-1}{r}u_r^\varepsilon$. Then there holds the following estimates (at least*

for sufficiently small $\varepsilon > 0$):

- (i) $\|u^\varepsilon\|_{C^0([0,R])} + \int_0^R |u_r^\varepsilon|^n dr \leq C_0,$
- (ii) $\|u^\varepsilon\|_{C^1([0,R])} + \|w^\varepsilon\|_{C^0([0,R])} \leq C_1,$
- (iii) $\|w_r^\varepsilon\|_{C^0([0,R])} \leq \frac{C_2}{\varepsilon},$
- (iv) $\|v^\varepsilon\|_{C^0([r_0,R])} \leq \frac{C_3}{\varepsilon r_0^{n-1}} \quad \forall 0 < r_0 \leq R,$
- (v) $\|v_r^\varepsilon\|_{C^0([r_0,R])} \leq \frac{C_4}{\varepsilon r_0^{(n-1)^2}} \quad \forall 0 < r_0 \leq R,$
- (vi) $\int_0^R |w_r^\varepsilon(r)|^2 dr + \int_0^R r^{2(n-1)} |v^\varepsilon(r)|^2 dr \leq \frac{C_5}{\varepsilon},$
- (vii) $\varepsilon \int_0^R r^{n-2-\alpha} |v^\varepsilon(r)|^2 dr + \int_0^R \frac{1}{r^\alpha} (u_r^\varepsilon(r))^n v^\varepsilon(r) dr \leq \frac{C_6}{\varepsilon} \quad \forall \alpha < n-1,$
- (viii) $\varepsilon \int_0^R r^{n-1} |v_r^\varepsilon(r)|^2 dr + \int_0^R (u_r^\varepsilon(r))^{n-1} |v^\varepsilon(r)|^2 dr \leq \frac{C_7}{\varepsilon} \quad \text{for } n \geq 3,$
- (ix) $\varepsilon \int_0^R r^{2-\alpha} |v_r^\varepsilon(r)|^2 dr + \int_0^R r^{1-\alpha} u_r^\varepsilon(r) |v^\varepsilon(r)|^2 dr \leq \frac{C_8}{\varepsilon} \quad \text{for } n = 2, \alpha < 1,$

where $C_j = C_j(R, f, n) > 0$ for $j = 0, 1, 2, \dots, 8$ are ε -independent positive constants.

PROOF. We divide the proof into five steps

Step 1: Since u^ε is monotone increasing,

$$(3.41) \quad \max_{[0,R]} u^\varepsilon(r) \leq u^\varepsilon(R) = g(R).$$

We note that the above estimate also follows from $\Delta u^\varepsilon \geq 0$ and the maximum principle.

On noting that w^ε satisfies equation (3.11), integrating (3.11) over $(0, R)$ and using integration by parts on the first term on the left-hand side yield

$$-\varepsilon w_r^\varepsilon(R) + \varepsilon(n-1) \int_0^R \frac{1}{r} w^\varepsilon(r) dr + \frac{1}{n} \int_0^R \left[\frac{w^\varepsilon(r)}{r^{n-1}} \right]^n dr = \int_0^R L_f(r) dr.$$

Because $w_r^\varepsilon(R) = \varepsilon R^{n-1}$ and $w^\varepsilon \geq 0$, the above equation and the relation $w^\varepsilon = r^{n-1} u_r^\varepsilon$ imply that

$$(3.42) \quad \int_0^R \left| \frac{w^\varepsilon(r)}{r^{n-1}} \right|^n dr = \int_0^R |u_r^\varepsilon(r)|^n dr \leq nR[L_f(R) + \varepsilon^2 R^{n-2}].$$

It then follows from (3.41), (3.42) and (3.36) that

$$(3.43) \quad g(R) - nR[L_f(R) + \varepsilon^2 R^{n-2}]^{\frac{1}{n}} \leq u^\varepsilon(r) \leq g(R) \quad \forall r \in [0, R].$$

Hence, u^ε is uniformly bounded (in ε) in $[0, R]$, and (i) holds.

Step 2: Let

$$v^\varepsilon := \Delta u^\varepsilon = u_{rr}^\varepsilon + \frac{n-1}{r} u_r^\varepsilon = \frac{1}{r^{n-1}} (r^{n-1} u_r^\varepsilon)_r.$$

By (3.6) we have

$$(3.44) \quad -\varepsilon(r^{n-1}v_r^\varepsilon)_r + \frac{1}{n}((u_r^\varepsilon)^n)_r = r^{n-1}f \quad \text{in } (0, R).$$

It was proved in the previous theorem that $v^\varepsilon > \varepsilon$ in $(\frac{R}{2}, R)$ for sufficiently small $\varepsilon > 0$ and it takes its minimum value ε at $r = R$. Hence we have $v_r^\varepsilon(R) \leq 0$.¹

Integrating (3.44) over $(0, R)$ yields

$$-\varepsilon r^{n-1}v_r^\varepsilon \Big|_{r=0}^{r=R} + \frac{1}{n}(u_r^\varepsilon)^n \Big|_{r=0}^{r=R} = L_f(R).$$

hence,

$$(u_r^\varepsilon(R))^n = nL_f(R) + \varepsilon nR^{n-1}v_r^\varepsilon(R) \leq nL_f(R),$$

therefore,

$$(3.45) \quad u_r^\varepsilon(R) = |u_r^\varepsilon(R)| \leq (nL_f(R))^{\frac{1}{n}}.$$

Here we have used boundary condition (3.8) and the fact that $v_r^\varepsilon(R) \leq 0$ and $u_r^\varepsilon \geq 0$.

By the definition of $w^\varepsilon(r) := r^{n-1}u_r^\varepsilon(r)$ we have

$$(3.46) \quad w^\varepsilon(R) = |w^\varepsilon(R)| \leq R^{n-1}|u_r^\varepsilon(R)| \leq R^{n-1}(nL_f(R))^{\frac{1}{n}}.$$

Using the identity

$$v^\varepsilon(r) = \Delta u^\varepsilon(r) = u_{rr}^\varepsilon(r) + \frac{n-1}{r}u_r^\varepsilon(r),$$

we get

$$u_{rr}^\varepsilon(R) = \Delta u^\varepsilon(R) - \frac{n-1}{R}u_r^\varepsilon(R) = \varepsilon - \frac{n-1}{R}u_r^\varepsilon(R).$$

Hence,

$$(3.47) \quad |u_{rr}^\varepsilon(R)| \leq \varepsilon + \frac{n-1}{R}|u_r^\varepsilon(R)| \leq \varepsilon + \frac{n-1}{R}(nL_f(R))^{\frac{1}{n}}.$$

Step 3: From Theorem 3.9 we have that $w_r^\varepsilon(r) \geq 0$ in $(0, R)$, and hence, w^ε is monotone increasing. Consequently,

$$(3.48) \quad \max_{[0, R]} w^\varepsilon(r) = \max_{[0, R]} |w^\varepsilon(r)| \leq w^\varepsilon(R) \leq R^{n-1}(nL_f(R))^{\frac{1}{n}}.$$

Evidently, (3.48) and the relation $w^\varepsilon(r) = r^{n-1}u_r^\varepsilon(r)$ as well as $\lim_{r \rightarrow 0^+} u_r^\varepsilon(r) = 0$ imply that there exists $r_0 > 0$ such that

$$(3.49) \quad \max_{[0, R]} u_r^\varepsilon(r) = \max_{[0, R]} |u_r^\varepsilon(r)| \leq \frac{1}{2} + \left(\frac{R}{r_0}\right)^{n-1}(nL_f(R))^{\frac{1}{n}}.$$

Hence, (ii) holds.

In addition, since w_r^ε satisfies the linear elliptic equation (3.38), by the pointwise estimate for linear elliptic equations [42, Theorem 3.7] we have

$$(3.50) \quad \max_{[0, R]} |w_r^\varepsilon(r)| \leq \varepsilon R^{n-1} + \frac{1}{\varepsilon} \left(R^{n-1} \|f\|_{L^\infty} + (n-1) \|r^{-1}L_f\|_\infty \right. \\ \left. + \frac{(n-1)^2}{n} \|r^{-1}(u_r^\varepsilon)^n\|_\infty \right).$$

¹This is the only place in the proof where we may need to require ε to be sufficiently small.

Since $w_r^\varepsilon = r^{n-1} \Delta u^\varepsilon =: r^{n-1} v^\varepsilon$, it follows from (3.50) that for any $r_0 > 0$ there holds

$$(3.51) \quad \begin{aligned} \max_{[r_0, R]} |v^\varepsilon(r)| &= \max_{[r_0, R]} |\Delta u^\varepsilon(r)| \\ &\leq \varepsilon \left(\frac{R}{r_0} \right)^{n-1} + \frac{1}{\varepsilon r_0^{n-1}} \left(R^{n-1} \|f\|_{L^\infty} + (n-1) \|r^{-1} L_f\|_\infty \right. \\ &\quad \left. + \frac{(n-1)^2}{n} \|r^{-1} (u_r^\varepsilon)^n\|_\infty \right). \end{aligned}$$

Thus, (iii) and (iv) are true.

Integrating (3.44) over $(0, r)$ yields

$$(3.52) \quad -\varepsilon r^{n-1} v_r^\varepsilon + \frac{1}{n} (u_r^\varepsilon)^n = L_f \quad \text{in } (0, R).$$

By (3.52) and (3.49) we conclude that for any $r_0 > 0$ there holds

$$(3.53) \quad \max_{[r_0, R]} |v_r^\varepsilon(r)| = \max_{[r_0, R]} |(\Delta u^\varepsilon(r))_r| \leq \frac{1}{\varepsilon} \left(1 + \left(\frac{R}{r_0} \right)^{n(n-1)} \right) \frac{L_f(R)}{r_0^{n-1}}.$$

So (v) holds.

Step 4: Testing (3.11) with w^ε and integrating by parts twice on the first term on the left-hand side, we get

$$\begin{aligned} & -\varepsilon^2 R^{n-1} w^\varepsilon(R) + \frac{\varepsilon}{2} \int_0^R |w_r^\varepsilon(r)|^2 dr + \frac{\varepsilon(n-1)}{2R} [w^\varepsilon(R)]^2 \\ & + \int_0^R \frac{\varepsilon(n-1)}{2r^2} |w^\varepsilon(r)|^2 dr + \int_0^R \frac{1}{nr^{n(n-1)}} |w^\varepsilon(r)|^{n+1} dr = \int_0^R L_f(r) w^\varepsilon(r) dr. \end{aligned}$$

Combing the above equation and (3.48) we obtain

$$(3.54) \quad \begin{aligned} & \frac{\varepsilon}{2} \int_0^R |w_r^\varepsilon(r)|^2 dr + \int_0^R \frac{\varepsilon(n-1)}{2r^2} |w^\varepsilon(r)|^2 dr + \int_0^R \frac{1}{nr^{n(n-1)}} |w^\varepsilon(r)|^{n+1} dr \\ & \leq R [\varepsilon^2 R^{n-2} + L_f(R)] (nL_f(R))^{\frac{1}{n}}. \end{aligned}$$

Consequently,

$$(3.55) \quad \begin{aligned} & \frac{\varepsilon}{2} \int_0^R |r^{n-1} \Delta u^\varepsilon(r)|^2 dr + \frac{\varepsilon(n-1)}{2} \int_0^R r^{2(n-2)} |u_r^\varepsilon(r)|^2 dr \\ & + \frac{1}{n} \int_0^R r^{n-1} |u_r^\varepsilon(r)|^{n+1} dr \leq R [\varepsilon^2 R^{n-2} + L_f(R)] (nL_f(R))^{\frac{1}{n}}. \end{aligned}$$

Hence, (vi) holds.

Step 5: For any real number $\alpha < n-1$, testing (3.52) with $r^{-\alpha} v^\varepsilon$ and using $v^\varepsilon(R) = \varepsilon$ we get

$$(3.56) \quad \begin{aligned} & -\frac{\varepsilon^3}{2} R^{n-1-\alpha} + \frac{\varepsilon(n-1-\alpha)}{2} \int_0^R r^{n-2-\alpha} |v^\varepsilon(r)|^2 dr \\ & + \int_0^R \frac{1}{nr^\alpha} (u_r^\varepsilon(r))^n v^\varepsilon(r) dr = \int_0^R \frac{1}{r^\alpha} L_f(r) v^\varepsilon(r) dr. \end{aligned}$$

On noting that $v^\varepsilon \geq 0$, $u_r^\varepsilon \geq 0$, and

$$L_f(r) = \int_0^r s^{n-1} f(s) ds \leq \frac{r^n}{n} \|f\|_{L^\infty},$$

it follows from (3.56) that

$$(3.57) \quad \frac{\varepsilon(n-1-\alpha)}{4} \int_0^R r^{n-2-\alpha} |v^\varepsilon(r)|^2 dr + \frac{1}{n} \int_0^R \frac{1}{r^\alpha} (u_r^\varepsilon(r))^n v^\varepsilon(r) dr \\ \leq \frac{\varepsilon^3}{2} R^{n-1-\alpha} + \frac{R^{n+3-\alpha} \|f\|_{L^\infty}^2}{\varepsilon n^2 (n-1-\alpha)(n+3-\alpha)} dr \quad \forall \alpha < n-1.$$

This gives (vii)

Recall that

$$v^\varepsilon := \Delta u^\varepsilon = u_{rr}^\varepsilon + \frac{n-1}{r} u_r^\varepsilon,$$

and therefore, we can rewrite (3.44) as follows

$$-\varepsilon (r^{n-1} v_r^\varepsilon)_r + (u_r^\varepsilon)^{n-1} v^\varepsilon = r^{n-1} f + \frac{n-1}{r} (u_r^\varepsilon)^n \quad \text{in } (0, R).$$

Testing the above equation with $r^\beta v^\varepsilon$ for $\beta > 1-n$ and using $v^\varepsilon(R) = \varepsilon$, we get

$$-\varepsilon^2 R^{n-1+\beta} v_r^\varepsilon(R) + \varepsilon \int_0^R r^{n-1+\beta} |v_r^\varepsilon(r)|^2 dr + \frac{\varepsilon^3 \beta R^{n+\beta-2}}{2} \\ - \frac{\varepsilon \beta (n+\beta-2)}{2} \int_0^R r^{n+\beta-3} |v^\varepsilon(r)|^2 dr + \int_0^R r^\beta (u_r^\varepsilon(r))^{n-1} |v^\varepsilon(r)|^2 dr \\ = \int_0^R \left[r^{n-1+\beta} f(r) + \frac{n-1}{r^{1-\beta}} (u_r^\varepsilon(r))^n \right] v^\varepsilon(r) dr.$$

Hence,

$$(3.58) \quad -\varepsilon^2 R^{n-1+\beta} v_r^\varepsilon(R) + \varepsilon \int_0^R r^{n-1+\beta} |v_r^\varepsilon(r)|^2 dr + \frac{\varepsilon^3 \beta R^{n+\beta-2}}{2} \\ - \frac{\varepsilon \beta (n+\beta-2)}{2} \int_0^R r^{n+\beta-3} |v^\varepsilon(r)|^2 dr + \int_0^R r^\beta (u_r^\varepsilon(r))^{n-1} |v^\varepsilon(r)|^2 dr \\ \leq \frac{\varepsilon}{2} \int_0^R r^{n-1+\beta} |v_r^\varepsilon(r)|^2 dr + \frac{1}{2\varepsilon} \int_0^R r^{n-1+\beta} |f(r)|^2 dr \\ + (n-1) \int_0^R \frac{1}{r^{1-\beta}} (u_r^\varepsilon(r))^n v^\varepsilon(r) dr.$$

To continue, we consider the cases $n=2$ and $n>2$ separately. First, for $n>2$, since $v_r^\varepsilon(R) \leq 0$, it follows from (3.57) with $\alpha=1$ and (3.58) with $\beta=0$ that

$$(3.59) \quad \frac{\varepsilon}{2} \int_0^R r^{n-1} |v_r^\varepsilon(r)|^2 dr + \int_0^R (u_r^\varepsilon(r))^{n-1} |v^\varepsilon(r)|^2 dr \\ \leq \frac{1}{2\varepsilon} \int_0^R r^{n-1} |f(r)|^2 dr + R^{n-2} \left[\frac{\varepsilon^3 n(n-1)}{2} + \frac{R^4 \|f\|_{L^\infty}^2}{\varepsilon(n^2-4)} dr \right].$$

When $n=2$, we note that $\alpha=1$ is not allowed in (3.57). Let $\alpha < 1$ be fixed in (3.57), set $\beta=1-\alpha$ in (3.58) we get

$$(3.60) \quad \frac{\varepsilon}{2} \int_0^R r^{2-\alpha} |v_r^\varepsilon(r)|^2 dr + \int_0^R r^{1-\alpha} u_r^\varepsilon(r) |v^\varepsilon(r)|^2 dr \\ \leq \frac{1}{2\varepsilon} \int_0^R r^{2-\alpha} |f(r)|^2 dr + 2R^{1-\alpha} \left[\varepsilon^3 + \frac{R^4 \|f\|_{L^\infty}^2}{\varepsilon(1-\alpha)(5-\alpha)} dr \right].$$

Hence, (viii) and (ix) hold. The proof is complete. \square

We now state and prove the following convexity result for the vanishing moment approximation u^ε .

Theorem 3.12. *Suppose $f \in C^0((0, R))$ and there exists a positive constant f_0 such that $f \geq f_0$ on $[0, R]$. Let u^ε denote the unique monotone increasing classical solution to problem (3.6)–(3.9).*

- (i) *If $n = 2, 3$, then either u^ε is strictly convex in $(0, R)$ or there exists an ε -independent positive constant c_0 such that u^ε is strictly convex in $(0, R - c_0\varepsilon)$.*
- (ii) *If $n > 3$, then there exists a monotone decreasing sequence $\{s_j\}_{j \geq 0} \subset (0, R)$ and two corresponding sequences $\{\varepsilon_j\}_{j \geq 0} \subset (0, 1)$, which is also monotone decreasing, and $\{r_j^*\}_{j \geq 0} \subset (0, R)$ satisfying $s_j \searrow 0^+$ as $j \rightarrow \infty$ and $u_{rr}^\varepsilon(s_j) \geq 0$ and $R - r_j^* = O(\varepsilon)$ such that for each $j \geq 0$, u^ε is strictly convex in (s_j, r_j^*) for all $\varepsilon \in (0, \varepsilon_j)$.*

PROOF. We divide the proof into three steps.

Step 1: Let $w^\varepsilon := r^{n-1}u_r^\varepsilon$ and $v^\varepsilon := \Delta u^\varepsilon = u_{rr}^\varepsilon + \frac{n-1}{r}u_r^\varepsilon = w_r^\varepsilon$ be same as before, and define $\eta^\varepsilon := r^{n-1}u_{rr}^\varepsilon$. On noting that

$$r^{n-1}v_r^\varepsilon = (r^{n-1}u_{rr}^\varepsilon)_r - (n-1)r^{n-3}u_r^\varepsilon = \eta_r^\varepsilon + \frac{1}{r}\eta^\varepsilon - r^{n-2}v^\varepsilon,$$

(3.44) can be rewritten as

$$(3.61) \quad -\varepsilon\eta_{rr}^\varepsilon + \left[\frac{2\varepsilon}{r^2} + \frac{(u_r^\varepsilon)^{n-1}}{r^{n-1}} \right] \eta^\varepsilon = r^{n-1}f + \varepsilon(3-n)r^{n-3}v^\varepsilon \quad \text{in } (0, R).$$

So η^ε satisfies a linear uniformly elliptic equation.

Clearly, $\eta^\varepsilon(0) = 0$. We claim that there exists (at least one) $r_1 \in (0, R]$ such that $\eta^\varepsilon(r_1) \geq 0$. If not, then $\eta^\varepsilon < 0$ in $(0, R]$, so is u_{rr}^ε . This implies that u_r^ε is monotone decreasing in $(0, R]$. Since $u_r^\varepsilon(0) = 0$, hence, $u_r^\varepsilon < 0$ in $(0, R]$. But this contradicts with the fact that $u_r^\varepsilon \geq 0$ in $(0, R]$. Therefore, the claim must be true.

Due to the factor $(3-n)$ in the second term on the right-hand side of (3.61), the situations for the cases $n \leq 3$ and $n > 3$ are different, and need to be handled slightly different.

Step 2: The case $n = 2, 3$. Since $v^\varepsilon \geq 0$, hence,

$$(3.62) \quad -\varepsilon\eta_{rr}^\varepsilon + \left[\frac{2\varepsilon}{r^2} + \frac{(u_r^\varepsilon)^{n-1}}{r^{n-1}} \right] \eta^\varepsilon \geq 0 \quad \text{in } (0, R).$$

Therefore, η^ε is a supersolution to a linear uniformly elliptic differential operator. By the weak maximum principle (cf. [32, page 329]) we have

$$\min_{[0, r_1]} \eta^\varepsilon(r) \geq \min\{0, \eta^\varepsilon(0), \eta^\varepsilon(r_1)\} = \min\{0, \eta^\varepsilon(r_1)\} = 0.$$

Let $r_* = \max\{r_1 \in (0, R]; \eta^\varepsilon(r_1) \geq 0\}$. By the above argument and the definition of r_* we have $\eta^\varepsilon \geq 0$ in $[0, r_*]$, $\eta^\varepsilon(r_*) = 0$ if $r_* \neq R$, and $\eta^\varepsilon < 0$ in $(r_*, R]$. If $r_* = R$, then $\eta^\varepsilon \geq 0$ in $[0, R]$. An application of the strong maximum principle to conclude that $\eta^\varepsilon > 0$ in $(0, R)$. Hence, $u_{rr}^\varepsilon > 0$ in $(0, R)$. Thus, u^ε is strictly convex in $(0, R)$. So the first part of the theorem's assertion is proved.

On the other hand, if $r_* < R$, we only know that u^ε is strictly convex in $(0, r_*)$. We now prove that $R - r_* = O(\varepsilon)$, which then justifies the remaining part of the theorem's assertion.

By (3.61) and the above setup we have

$$-\varepsilon\eta_{rr}^\varepsilon \geq r^{n-1}f \geq f_0r^{n-1} \quad \text{in } (r_*, R).$$

Integrating the above inequality over (r_*, r) for $r \leq R$ and noting that $\eta_r^\varepsilon(r_*) \leq 0$ we get

$$-\varepsilon\eta_r^\varepsilon \geq \frac{f_0}{n}(r^n - r_*^n) \quad \text{in } (r_*, R).$$

Integrating again over (r_*, R) and using the fact that $\eta^\varepsilon(r_*) = 0$ and the following algebraic inequality

$$\frac{1}{n+1} \frac{R^{n+1} - r_*^{n+1}}{R - r_*} - r_*^n \geq \frac{1}{n+1} R^n$$

we arrive at

$$-\varepsilon R^{n-1}u_{rr}(R) = -\varepsilon\eta^\varepsilon(R) \geq \frac{f_0 R^n (R - r_*)}{n(n+1)}.$$

It follows from (3.47) that

$$\begin{aligned} R - r_* &\leq \frac{\varepsilon n(n+1)|u_{rr}(R)|}{Rf_0} \\ &\leq \frac{\varepsilon n(n+1)}{R^2 f_0} \left[\varepsilon R + (n-1)(nL_f(R))^{\frac{1}{n}} \right] =: c_0 \varepsilon. \end{aligned}$$

Thus,

$$(3.63) \quad R - r_* = O(\varepsilon),$$

and u^ε is strictly convex in $(0, R - c_0 \varepsilon)$.

Step 3: The case $n > 3$: First, By the argument used in *Step 1*, it is easy to show that η^ε can not be strictly negative in the whole of any neighborhood of $r = 0$. Thus, there exists a monotone decreasing sequence $\{s_j\}_{j \geq 0} \subset (0, R)$ such that $s_j \searrow 0^+$ as $j \rightarrow \infty$ and $\eta^\varepsilon(s_j) \geq 0$.

Second, we note that

$$\begin{aligned} \varepsilon(3-n)r^{n-3}v^\varepsilon &= \varepsilon(3-n)r^{n-3} \left(u_{rr}^\varepsilon + \frac{n-1}{r}u_r^\varepsilon \right) \\ &= \varepsilon(3-n) \left(\frac{\eta^\varepsilon}{r^2} + (n-1)r^{n-4}u_r^\varepsilon \right) \end{aligned}$$

Using this identity in (3.61), we have

$$-\varepsilon\eta_{rr}^\varepsilon + \left[\frac{(n-1)\varepsilon}{r^2} + \frac{(u_r^\varepsilon)^{n-1}}{r^{n-1}} \right] \eta^\varepsilon = r^{n-4} [r^3 f + \varepsilon(3-n)(n-1)u_r^\varepsilon] \quad \text{in } (0, R).$$

By (ii) of Theorem 3.11 we know that u_r^ε is uniformly bounded in $[0, R]$. Then for each s_j there exists an $\varepsilon_j > 0$ (without loss of the generality, choose $\varepsilon_j < \varepsilon_{j-1}$) such that for $\varepsilon \in (0, \varepsilon_j)$

$$[r^3 f + \varepsilon(3-n)(n-1)u_r^\varepsilon] \geq 0 \quad \text{in } (s_j, R).$$

Hence, η^ε is a supersolution to a linear uniformly elliptic operator on (s_j, R) for $\varepsilon < \varepsilon_j$.

Third, for each fixed $j \geq 1$, let $r_j^* = \max\{r \in (s_j, R]; \eta^\varepsilon(r) \geq 0\}$. Trivially, by the construction, $r_j^* \geq s_{j-1} > s_j$. By the weak maximum principle (cf. [32, page 329]) we have

$$\min_{[s_j, r_j^*]} \eta^\varepsilon(r) \geq \min\{0, \eta^\varepsilon(s_j), \eta^\varepsilon(r_j^*)\} \geq 0.$$

Finally, repeating the argument of *Step 2*., we conclude that u^ε is either strictly convex in (s_j, R) or in (s_j, r_j^*) with $R - r_j^* = O(\varepsilon)$ for $\varepsilon \in (0, \varepsilon_j)$. The proof is now complete. \square

3.4. Convergence of vanishing moment approximations

The goal of this section is to show that the solution u^ε of problem (3.6)–(3.9) converges to the convex solution u of problem (3.1)–(3.3). We present two different proofs for the convergence. The first proof is based on the variational formulations of both problems. The second proof, which can be extended to more general non-radially symmetric case [36], is done in the viscosity solution setting [26]. Both proofs mainly rely on two key ingredients. The first is the solution estimates obtained in Theorem 3.11, the second is the uniqueness of solutions to problem (3.1)–(3.3).

Theorem 3.13. *Suppose $f \in C^0((0, R))$ and there exists a positive constant f_0 such that $f \geq f_0$ in $[0, R]$. Let u denote the convex (classical) solution to problem (3.1)–(3.3) and u^ε be the monotone increasing classical solution to problem (3.6)–(3.9). Then*

- (i) $u^0 = \lim_{\varepsilon \rightarrow 0^+} u^\varepsilon$ exists pointwise and u^ε converges to u^0 uniformly in every compact subset of $(0, R)$ as $\varepsilon \rightarrow 0^+$. Moreover, u^0 is strictly convex in every compact subset, hence, it is strictly convex in $[0, R]$.
- (ii) u_r^ε converges to u_r^0 weakly $*$ in $L^\infty((0, R))$ as $\varepsilon \rightarrow 0^+$.
- (iii) $u^0 \equiv u$.

PROOF. It follows from (ii) of Theorem 3.11 that $\|u^\varepsilon\|_{C^1([0, R])}$ is uniformly bounded in ε , then $\{u^\varepsilon\}_{\varepsilon \geq 0}$ is uniformly equicontinuous. By Arzela-Ascoli compactness theorem (cf. [32, page 635]) we conclude that there exists a subsequence of $\{u^\varepsilon\}_{\varepsilon \geq 0}$ (still denoted by the same notation) and $u^0 \in C^1([0, R])$ such that

$$\begin{aligned} u^\varepsilon &\longrightarrow u^0 && \text{uniformly in every compact set } E \subset (0, R) \text{ as } \varepsilon \rightarrow 0^+, \\ u_r^\varepsilon &\longrightarrow u_r^0 && \text{weakly } * \text{ in } L^\infty((0, R)) \text{ as } \varepsilon \rightarrow 0^+, \end{aligned}$$

and $u^\varepsilon(R) = g(R)$ implies that $u^0(R) = g(R)$.

In addition, by Theorem 3.12 we have that for every compact subset $E \subset (0, R)$ there exists $\varepsilon_0 > 0$ such that $E \subset (0, R - c_0\varepsilon)$ and u^ε is strictly convex in $(0, R - c_0\varepsilon)$ for $\varepsilon < \varepsilon_0$. It follows from a well-known property of convex functions (cf. [48]) that u^0 must be strictly convex in E and $u^0 \in C_{\text{loc}}^{1,1}((0, R))$.

Testing equation (3.44) with an arbitrary function $\chi \in C_0^2((0, R))$ yields

$$(3.64) \quad \varepsilon \int_0^R r^{n-1} v_r^\varepsilon(r) \chi_r(r) dr - \frac{1}{n} \int_0^R (u_r^\varepsilon(r))^n \chi_r(r) dr = \int_0^R r^{n-1} f(r) \chi(r) dr,$$

where as before $v^\varepsilon = \Delta u^\varepsilon = u_{rr}^\varepsilon + \frac{n-1}{r} u_r^\varepsilon$.

By Schwartz inequality and (vi) of Theorem 3.11 we have

$$\begin{aligned} \varepsilon \int_0^R r^{n-1} v_r^\varepsilon(r) \chi_r(r) dr &= -\varepsilon \int_0^R r^{n-1} v^\varepsilon(r) \left[\chi_{rr}(r) + \frac{n-1}{r} \chi_r(r) \right] dr \\ &\leq \varepsilon \left(\int_0^R r^{2(n-1)} |v^\varepsilon(r)|^2 dr \right)^{\frac{1}{2}} \left(\int_0^R |\Delta \chi(r)|^2 dr \right)^{\frac{1}{2}} \\ &\leq \sqrt{\varepsilon} C_5 \left(\int_0^R |\Delta \chi(r)|^2 dr \right)^{\frac{1}{2}} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+. \end{aligned}$$

Setting $\varepsilon \rightarrow 0^+$ in (3.64) and using the Lebesgue Dominated Convergence Theorem yield

$$(3.65) \quad -\frac{1}{n} \int_0^R (u_r^0(r))^n \chi_r(r) dr = \int_0^R r^{n-1} f(r) \chi(r) dr \quad \forall \chi \in C_0^1((0, R)).$$

It also follows from a standard test function argument that

$$u_r^0(0) = 0.$$

This means that $u^0 \in C^1([0, R]) \cap C_{\text{loc}}^{1,1}((0, R))$ is a convex weak solution to problem (3.1)–(3.3). By the uniqueness of convex solutions of problem (3.1)–(3.3), there must hold $u^0 \equiv u$.

Finally, since we have proved that every convergent subsequence of $\{u^\varepsilon\}_{\varepsilon \geq 0}$ converges to the unique convex classical solution u of problem (3.1)–(3.3), the whole sequence $\{u^\varepsilon\}_{\varepsilon \geq 0}$ must converge to u . The proof is complete. \square

Next, we state and prove a different version of Theorem 3.13. The difference is that we now only assume problem (3.1)–(3.3) has a unique strictly convex viscosity solution and so the proof must be adapted to the viscosity solution framework.

Theorem 3.14. *Suppose $f \in C^0((0, R))$ and there exists a positive constant f_0 such that $f \geq f_0$ on $[0, R]$. Let u denote the strictly convex viscosity solution to problem (3.1)–(3.3) and u^ε be the monotone increasing classical solution to problem (3.6)–(3.9). Then the statements of Theorem 3.13 still hold.*

PROOF. Except the step of proving the variational formulation (3.65), all other parts of the proof of Theorem 3.13 are still valid. So we only need to show that u^0 is a viscosity solution of problem (3.1)–(3.3), which is verified below by the definition of viscosity solutions.

Let $\phi \in C^2([0, R])$ be strictly convex², and suppose that $u^0 - \phi$ has a local maximum at a point $r_0 \in (0, R)$, that is, there exists a (small) number $\delta_0 > 0$ such that $(r_0 - \delta_0, r_0 + \delta_0) \subset \subset (0, R)$ and

$$u^0(r) - \phi(r) \leq u^0(r_0) - \phi(r_0) \quad \forall r \in (r_0 - \delta_0, r_0 + \delta_0).$$

Since u^ε (which still denotes a subsequence) converges to u^0 uniformly in $[r_0 - \delta_0, r_0 + \delta_0]$, then for sufficiently small $\varepsilon > 0$, there exists $r_\varepsilon \in (0, R)$ such that $r_\varepsilon \rightarrow r_0$ as $\varepsilon \rightarrow 0^+$ and $u^\varepsilon - \phi$ has a local maximum at r_ε (see [32, Chapter 10] for a proof of the claim). By elementary calculus, we have

$$u_r^\varepsilon(r_\varepsilon) = \phi_r(r_\varepsilon), \quad u_{rr}^\varepsilon(r_\varepsilon) \leq \phi_{rr}(r_\varepsilon).$$

²In fact, ϕ can be taken as a convex quadratic polynomial (cf. [18, 44]).

Because both u^ε and ϕ are strictly convex, there exists a (small) constant $\rho_0 > 0$ such that for sufficiently small $\varepsilon > 0$

$$u_{rr}^\varepsilon(r) \leq \phi_{rr}(r) \quad \forall r \in (r_0 - \rho, r_0 + \rho), \quad \rho < \rho_0,$$

which together with an application of Taylor's formula implies that

$$u_r^\varepsilon(r) = \phi_r(r) + O(|r - r_\varepsilon|) \quad \forall r \in (r_0 - \rho, r_0 + \rho), \quad \rho < \rho_0.$$

Let $\chi \in C_0^2((r_0 - \rho, r_0 + \rho))$ with $\chi \geq 0$ and $\chi(r_0) > 0$. Testing (3.44) with χ yields

$$\begin{aligned} (3.66) \quad & \frac{1}{2n\rho} \int_{r_0-\rho}^{r_0+\rho} ((\phi_r(r))^n)_r \chi(r) dr = \frac{1}{2\rho} \int_{r_0-\rho}^{r_0+\rho} (\phi_r(r))^{n-1} \phi_{rr}(r) \chi(r) dr \\ & \geq \frac{1}{2\rho} \int_{r_0-\rho}^{r_0+\rho} [(u_r^\varepsilon(r))^{n-1} + O(|r - r_\varepsilon|^{n-1})] u_{rr}^\varepsilon(r) \chi(r) dr \\ & = \frac{1}{2n\rho} \int_{r_0-\rho}^{r_0+\rho} [((u_r^\varepsilon(r))^n)_r + O(|r - r_\varepsilon|^{n-1}) u_{rr}^\varepsilon(r)] \chi(r) dr \\ & \geq \frac{1}{2\rho} \int_{r_0-\rho}^{r_0+\rho} r^{n-1} [f(r) \chi(r) + \varepsilon v_r^\varepsilon(r) \chi_r(r)] dr \\ & \quad - C_9 \rho^{n-2} \int_{r_0-\rho}^{r_0+\rho} u_r^\varepsilon(r) \chi_r(r) dr \end{aligned}$$

for some positive ρ -independent constant C_9 . Here we have used the fact that $u_{rr}^\varepsilon \geq 0, \chi \geq 0$ in $[r_0 - \rho, r_0 + \rho]$ to get the last inequality.

From (vi) of Theorem 3.11, we have

$$\begin{aligned} (3.67) \quad & \varepsilon \int_{r_0-\rho}^{r_0+\rho} r^{n-1} v_r^\varepsilon(r) \chi_r(r) dr \\ & = -\varepsilon \int_{r_0-\rho}^{r_0+\rho} r^{n-1} v^\varepsilon(r) \left[\chi_{rr}(r) + \frac{n-1}{r} \chi_r(r) \right] dr \\ & \leq \varepsilon \left(\int_0^R r^{2(n-1)} |v^\varepsilon(r)|^2 dr \right)^{\frac{1}{2}} \left(\int_0^R |\Delta \chi(r)|^2 dr \right)^{\frac{1}{2}} \\ & \leq \sqrt{\varepsilon C_5} \left(\int_0^R |\Delta \chi(r)|^2 dr \right)^{\frac{1}{2}}. \end{aligned}$$

Setting $\varepsilon \rightarrow 0^+$ in (3.66) and using (3.67) we get

$$\begin{aligned} (3.68) \quad & \frac{1}{2\rho} \int_{r_0-\rho}^{r_0+\rho} (\phi_r(r))^{n-1} \phi_{rr}(r) \chi(r) dr \\ & \geq \frac{1}{2\rho} \int_{r_0-\rho}^{r_0+\rho} r^{n-1} f(r) \chi(r) dr - C_9 \rho^{n-2} \int_{r_0-\rho}^{r_0+\rho} u_r^0(r) \chi_r(r) dr. \end{aligned}$$

Where we have used the fact that u_r^ε converges to u_r^0 weakly $*$ in $L^\infty((0, R))$ to pass to the limit in the last term on the right-hand side.

Finally, letting $\rho \rightarrow 0^+$ in (3.68) and using the Lebesgue-Besicovitch Differentiation Theorem (cf. [32]) we have

$$(\phi_r(r_0))^{n-1} \phi_{rr}(r_0) \chi(r_0) \geq r_0^{n-1} f(r_0) \chi(r_0).$$

Hence,

$$\left[\frac{\phi_r(r_0)}{r_0} \right]^{n-1} \phi_{rr}(r_0) \geq f(r_0),$$

so u^0 is a viscosity subsolution to equation (3.1).

Similarly, we can show that if $u^0 - \phi$ assumes a local minimum at $r_0 \in (0, R)$ for a strictly convex function $\phi \in C_0^2((0, R))$, there holds

$$\left[\frac{\phi_r(r_0)}{r_0} \right]^{n-1} \phi_{rr}(r_0) \leq f(r_0),$$

so u^0 is also a viscosity supersolution to equation (3.1). Thus, it is a viscosity solution. The proof is complete. \square

3.5. Rates of convergence

In this section, we derive rates of convergence for u^ε in various norms. Here we consider two cases, namely, the n -dimensional radially symmetric case and the general n -dimensional case, under different assumptions. In both cases, the linearization of the Monge-Ampère operator is explicitly exploited, and it plays a key role in our proofs.

Theorem 3.15. *Let u denote the strictly convex classical solution to problem (3.1)–(3.3) and u^ε be the monotone increasing classical solution to problem (3.6)–(3.9). Then there holds the following estimates:*

$$(3.69) \quad \left(\int_0^R \theta^\varepsilon(r) |u_r(r) - u_r^\varepsilon(r)|^2 dr \right)^{\frac{1}{2}} \leq \varepsilon^{\frac{3}{4}} C_{10},$$

$$(3.70) \quad \left(\int_0^R r^{n-1} |\Delta u(r) - \Delta u^\varepsilon(r)|^2 dr \right)^{\frac{1}{2}} \leq \varepsilon^{\frac{1}{4}} C_{11},$$

where $C_j = C_j(\|r^{n-1} \Delta u_r\|_{L^2})$ for $j = 10, 11$ are two positive ε -independent constants, and

$$(3.71) \quad \theta^\varepsilon(r) := \frac{(u_r)^n - (u_r^\varepsilon)^n}{u - u^\varepsilon} = \sum_{j=0}^{n-1} (u_r(r))^j (u_r^\varepsilon(r))^{n-1-j} > 0 \quad \text{in } (0, R].$$

PROOF. Let

$$v := \Delta u = u_{rr} - \frac{n-1}{r} u_r, \quad v^\varepsilon := \Delta u^\varepsilon = u_{rr}^\varepsilon - \frac{n-1}{r} u_r^\varepsilon, \quad e^\varepsilon := u - u^\varepsilon.$$

On noting that (3.6) can be written into (3.44), multiplying (3.1) by r^{n-1} and subtracting the resulting equation from (3.44) yield the following error equation:

$$(3.72) \quad \varepsilon (r^{n-1} v_r^\varepsilon)_r + \frac{1}{n} [(u_r)^n - (u_r^\varepsilon)^n]_r = 0 \quad \text{in } (0, R).$$

Testing (3.72) with e^ε using boundary condition $e_r^\varepsilon(0) = e^\varepsilon(R) = 0$ we get

$$(3.73) \quad \varepsilon \int_0^R r^{n-1} v_r^\varepsilon(r) e_r^\varepsilon(r) dr + \frac{1}{n} \int_0^R \theta^\varepsilon(r) |e_r^\varepsilon(r)|^2 dr = 0,$$

where θ^ε is defined by (3.71).

Integrating by parts on the first term of (3.73) and rearranging terms we get

$$(3.74) \quad \varepsilon \int_0^R r^{n-1} |\Delta e^\varepsilon(r)|^2 dr + \frac{1}{n} \int_0^R \theta^\varepsilon(r) |e_r^\varepsilon(r)|^2 dr \\ = \varepsilon R^{n-1} \Delta e^\varepsilon(R) e_r^\varepsilon(R) - \varepsilon \int_0^R r^{n-1} v_r(r) e_r^\varepsilon(r) dr.$$

We now bound the two terms on the right-hand side as follows. First, for the second term, a simple application of the Schwarz and Young's inequalities gives

$$(3.75) \quad \varepsilon \int_0^R r^{n-1} v_r(r) e_r^\varepsilon(r) dr \leq \frac{1}{4n} \int_0^R \theta^\varepsilon(r) |e_r^\varepsilon(r)|^2 dr \\ + \varepsilon^2 n \int_0^R \frac{r^{2(n-1)}}{\theta^\varepsilon(r)} |v_r(r)|^2 dr.$$

Second, to bound the first term on the right-hand side of (3.74), we use the boundary condition $v^\varepsilon(R) = \varepsilon$ to get

$$|\Delta e^\varepsilon(R)| = |v(R) - v^\varepsilon(R)| = |v(R) - \varepsilon| \leq |v(R)| + 1 =: M,$$

and

$$|R^{n-1} e_r^\varepsilon(R)|^2 = \int_0^R ((r^{n-1} e_r^\varepsilon(r))^2)_r dr = 2 \int_0^R r^{2(n-1)} e_r^\varepsilon(r) \Delta e^\varepsilon(r) dr \\ \leq 2 \left(\int_0^R r^{n-1} |\Delta e^\varepsilon(r)|^2 dr \right)^{\frac{1}{2}} \left(\int_0^R r^{3(n-1)} |e_r^\varepsilon(r)|^2 dr \right)^{\frac{1}{2}}.$$

Hence by Young's inequality, we get

$$(3.76) \quad |\varepsilon R^{n-1} \Delta e^\varepsilon(R) e_r^\varepsilon(R)| \\ \leq \sqrt{2} \varepsilon M \left(\int_0^R r^{n-1} |\Delta e^\varepsilon(r)|^2 dr \right)^{\frac{1}{4}} \left(\int_0^R r^{3(n-1)} |e_r^\varepsilon(r)|^2 dr \right)^{\frac{1}{4}} \\ \leq \frac{\varepsilon}{2} \int_0^R r^{n-1} |\Delta e^\varepsilon(r)|^2 dr + 2\varepsilon M^{\frac{4}{3}} \left(\int_0^R r^{3(n-1)} |e_r^\varepsilon(r)|^2 dr \right)^{\frac{1}{3}} \\ \leq \frac{\varepsilon}{2} \int_0^R r^{n-1} |\Delta e^\varepsilon(r)|^2 dr + \frac{1}{4n} \int_0^R \theta^\varepsilon(r) |e_r^\varepsilon(r)|^2 dr + \varepsilon^{\frac{3}{2}} n M^2 C$$

for some ε -independent constant $C = C(f, R, n) > 0$.

Combining (3.74)–(3.76) yields

$$(3.77) \quad \varepsilon \int_0^R r^{n-1} |\Delta e^\varepsilon(r)|^2 dr + \frac{1}{n} \int_0^R \theta^\varepsilon(r) |e_r^\varepsilon(r)|^2 dr \\ \leq 2\varepsilon^2 n \int_0^R \frac{r^{2(n-1)}}{\theta^\varepsilon(r)} |v_r(r)|^2 dr + \varepsilon^{\frac{3}{2}} n M^2 C.$$

Thus, (3.69) and (3.70) follow from the fact that $\|r^{n-1}(\theta^\varepsilon)^{-1}\|_{L^\infty} < \infty$. \square

Corollary 3.16. *Inequality (3.69) implies that there exists an ε -independent constant $C > 0$ such that*

$$(3.78) \quad \left(\int_0^R r^{n-1} |u_r(r) - u_r^\varepsilon(r)|^2 dr \right)^{\frac{1}{2}} \leq \varepsilon^{\frac{3}{4}} C C_{10}.$$

Since the proof is simple, we omit it.

Theorem 3.17. *Under the assumptions of Theorem 3.15, there also holds the following estimate:*

$$(3.79) \quad \left(\int_0^R r^{n-1} |u(r) - u^\varepsilon(r)|^2 dr \right)^{\frac{1}{2}} \leq \varepsilon C_{12}$$

for some positive ε -independent constant $C_{12} = C_{12}(R, n, u, C_{11})$.

PROOF. Let θ^ε be defined by (3.71), and e^ε , v and v^ε be same as in Theorem 3.15. Consider the following auxiliary problem:

$$(3.80) \quad (\theta^\varepsilon \phi_r)_r = nr^{n-1} e^\varepsilon \quad \text{in } (0, R),$$

$$(3.81) \quad \phi(R) = 0,$$

$$(3.82) \quad \phi_r(0) = 0.$$

We note that the left-hand side of (3.80) is the linearization of (3.1) at θ^ε .

Since $\theta^\varepsilon > 0$ in $(0, R]$, then (3.80) is a linear elliptic equation. Using the fact that $c_1 \geq r^{n-1}(\theta^\varepsilon)^{-1} \geq c_0 > 0$ in $[0, R]$ for some ε -independent positive constants c_0 and c_1 , it is easy to check that problem (3.80)–(3.82) has a unique classical solution ϕ . Moreover,

$$(3.83) \quad \int_0^R r^{n-1} |\phi_{rr}(r)|^2 dr + \int_0^R r^{n-1} |\phi_r(r)|^2 dr \leq \hat{C} \int_0^R r^{n-1} |e^\varepsilon(r)|^2 dr$$

for some ε -independent constant $\hat{C} = \hat{C}(f, R, n, c_0, c_1) > 0$.

Testing (3.80) by e^ε , using the facts that $\phi_r(0) = \phi(R) = 0$, $e^\varepsilon(R) = 0$ and $v^\varepsilon(R) = \varepsilon$ as well as error equation (3.72) we get

$$(3.84) \quad \begin{aligned} \int_0^R r^{n-1} |e^\varepsilon(r)|^2 dr &= -\frac{1}{n} \int_0^R \theta^\varepsilon(r) \phi_r(r) e_r^\varepsilon(r) dr \\ &= \varepsilon \int_0^R r^{n-1} v_r^\varepsilon(r) \phi_r(r) dr \\ &= \varepsilon R^{n-1} v^\varepsilon(R) \phi_r(R) - \varepsilon \int_0^R r^{n-1} v^\varepsilon(r) \Delta \phi(r) dr \\ &= \varepsilon^2 R^{n-1} \phi_r(R) + \varepsilon \int_0^R r^{n-1} [v(r) - v^\varepsilon(r)] \Delta \phi(r) dr \\ &\quad - \varepsilon \int_0^R r^{n-1} v(r) \Delta \phi(r) dr, \end{aligned}$$

where we have used the short-hand notation $\Delta \phi = r^{n-1}[\phi_{rr} + (n-1)r^{-1}\phi_r]$.

For each term on the right-hand side of (3.84) we have the following estimates:

$$\begin{aligned}
\varepsilon^2 R^{n-1} \phi_r(R) &= \frac{\varepsilon^2}{R} \int_0^R (r^n \phi_r(r))_r dr \\
&\leq \varepsilon^2 R^{\frac{n}{2}} \left(\int_0^R r^{n-1} |\phi_{rr}(r)|^2 dr \right)^{\frac{1}{2}} + \varepsilon^2 \sqrt{n} R^{\frac{n-2}{2}} \left(\int_0^R r^{n-1} |\phi_r(r)|^2 dr \right)^{\frac{1}{2}}, \\
\varepsilon \int_0^R r^{n-1} [v(r) - v^\varepsilon(r)] \Delta \phi(r) dr \\
&\leq \varepsilon \left(\int_0^R r^{n-1} |v(r) - v^\varepsilon(r)|^2 dr \right)^{\frac{1}{2}} \left(\int_0^R r^{n-1} |\Delta \phi(r)|^2 dr \right)^{\frac{1}{2}} \\
&\leq \varepsilon^{\frac{5}{4}} C_{11} \left(\int_0^R r^{n-1} |\Delta \phi(r)|^2 dr \right)^{\frac{1}{2}}, \\
- \varepsilon \int_0^R r^{n-1} v(r) \Delta \phi(r) dr &\leq \varepsilon \left(\int_0^R r^{n-1} |v(r)|^2 dr \right)^{\frac{1}{2}} \left(\int_0^R r^{n-1} |\Delta \phi(r)|^2 dr \right)^{\frac{1}{2}}.
\end{aligned}$$

Substituting the above estimates into (3.84) and using (3.83) we get

$$\begin{aligned}
(3.85) \quad &\int_0^R r^{n-1} |e^\varepsilon(r)|^2 dr \\
&\leq \varepsilon^2 R^{\frac{n}{2}} \left\{ \left(\int_0^R r^{n-1} |\phi_{rr}(r)|^2 dr \right)^{\frac{1}{2}} + \frac{\sqrt{n}}{R} \left(\int_0^R r^{n-1} |\phi_r(r)|^2 dr \right)^{\frac{1}{2}} \right\} \\
&\quad + \varepsilon (\varepsilon^{\frac{1}{4}} C_{11} + C_u) \left(\int_0^R r^{n-1} |\Delta \phi(r)|^2 dr \right)^{\frac{1}{2}} \\
&\leq 4\varepsilon (\varepsilon R^{\frac{n}{2}} + \varepsilon \sqrt{n} R^{-1} + \varepsilon^{\frac{1}{4}} C_{11} + C_u) \hat{C} \left(\int_0^R r^{n-1} |e^\varepsilon(r)|^2 dr \right)^{\frac{1}{2}}
\end{aligned}$$

for some ε -independent constant $C_u = C(u) > 0$.

Hence, by (3.85) we conclude that (3.79) holds with $C_{12} = 4(\varepsilon R^{\frac{n}{2}} + \varepsilon \sqrt{n} R^{-1} + \varepsilon^{\frac{1}{4}} C_{11} + C_u) \hat{C}$. The proof is complete. \square

Remark 3.18. The argument used in the above proof is so-called duality argument, which has been the main technique and used extensively in the finite element error analysis to derive error bounds in norms lower than the energy norm of the underlying PDE problem (cf. [13, 22] and the references therein). However, as far as we know, the duality argument is rarely (maybe has never been) used to derive error estimates in a PDE setting as done in the above proof.

Since the proofs of Theorem 3.15 and 3.17 only rely on the ellipticity of the linearization of the Monge-Ampère operator, hence, the results of both theorems can be easily extended to the general Monge-Ampère problem (1.11)–(1.12) and its moment approximation (2.9)–(2.11)₁³.

Theorem 3.19. *Let u denote the strictly convex viscosity solution to problem (1.11)–(1.12) and u^ε be a classical solution to problem (2.9)–(2.11)₁. Assume $u \in W^{2,\infty}(\Omega) \cap H^3(\Omega)$ and u^ε is either convex or “almost convex”⁴ in Ω . Then*

³This observation was pointed out to the first author by Professor Haijun Wu of Nanjing University, China, and the proof for (3.86) and (3.87) is essentially due to him.

⁴“Almost convex” means that u^ε is convex in Ω minus an ε -neighborhood of $\partial\Omega$, see Theorem 3.12 for a precise description.

there holds the following estimates:

$$(3.86) \quad \left(\int_{\Omega} |\nabla u - \nabla u^{\varepsilon}|^2 dx \right)^{\frac{1}{2}} \leq \varepsilon^{\frac{3}{4}} C_{13},$$

$$(3.87) \quad \left(\int_{\Omega} |\Delta u - \Delta u^{\varepsilon}|^2 dx \right)^{\frac{1}{2}} \leq \varepsilon^{\frac{1}{4}} C_{14},$$

$$(3.88) \quad \left(\int_{\Omega} |u - u^{\varepsilon}|^2 dx \right)^{\frac{1}{2}} \leq \varepsilon C_{15},$$

where $C_j = C_j(\|\nabla \Delta u\|_{L^2})$ for $j = 13, 14, 15$ are positive ε -independent constants.

PROOF. Since the proof follows the exact same lines as those for Theorem 3.15, we just briefly highlight the main steps.

First, the error equation (3.72) is replaced by

$$(3.89) \quad \varepsilon \Delta v^{\varepsilon} + \det(D^2 u) - \det(D^2 u^{\varepsilon}) = 0 \quad \text{in } \Omega,$$

where $v^{\varepsilon} = \Delta u^{\varepsilon}$.

Next, equation (3.71) becomes

$$(3.90) \quad \begin{aligned} \varepsilon \int_{\Omega} |\Delta e^{\varepsilon}|^2 dx + \int_{\Omega} \theta^{\varepsilon} \nabla e^{\varepsilon} \cdot \nabla e^{\varepsilon} dx \\ = \int_{\partial\Omega} \Delta e^{\varepsilon} \frac{\partial e^{\varepsilon}}{\partial \nu} dS - \varepsilon \int_{\Omega} \nabla v \cdot \nabla e^{\varepsilon} dx, \end{aligned}$$

where

$$(3.91) \quad \theta^{\varepsilon} = \Phi^{\varepsilon} := \text{cof}(tD^2 u + (1-t)D^2 u^{\varepsilon}) \quad \text{for some } t \in [0, 1],$$

now stands for the cofactor matrix of $tD^2 u + (1-t)D^2 u^{\varepsilon}$. Since u is assumed to be strictly convex and u^{ε} is “almost convex”, then there exists a positive constant θ_0 such that (see Chapter 4)

$$\theta^{\varepsilon} \nabla e^{\varepsilon} \cdot \nabla e^{\varepsilon} \geq \theta_0 |\nabla e^{\varepsilon}|^2.$$

It remains to derive a boundary estimate that is analogous to (3.76). To the end, by the boundary condition $v^{\varepsilon}|_{\partial\Omega} = \varepsilon$ and the trace inequality we have

$$(3.92) \quad \begin{aligned} \int_{\partial\Omega} \Delta e^{\varepsilon} \frac{\partial e^{\varepsilon}}{\partial \nu} dS &\leq \varepsilon (\varepsilon |\partial\Omega| + \|\Delta u\|_{L^2(\partial\Omega)}^2)^{\frac{1}{2}} \left\| \frac{\partial e^{\varepsilon}}{\partial \nu} \right\|_{L^2(\partial\Omega)} \\ &\leq \varepsilon M \|\nabla e^{\varepsilon}\|_{L^2(\Omega)}^{\frac{1}{2}} \|\Delta e^{\varepsilon}\|_{L^2(\Omega)}^{\frac{1}{2}} \\ &\leq \frac{\varepsilon}{2} \|\Delta e^{\varepsilon}\|_{L^2(\Omega)}^2 + M^{\frac{4}{3}} \varepsilon \|\nabla e^{\varepsilon}\|_{L^2(\Omega)}^{\frac{2}{3}} \\ &\leq \frac{\varepsilon}{2} \|\Delta e^{\varepsilon}\|_{L^2(\Omega)}^2 + \frac{\theta_0}{4} \|\nabla e^{\varepsilon}\|_{L^2(\Omega)}^2 + \frac{\varepsilon^{\frac{3}{2}} M^2}{\theta_0}. \end{aligned}$$

The desired estimates (3.86) and (3.87) follow from combining (3.90) and (3.92).

Finally, (3.88) can be derived by using the same duality argument as that used in the proof of Theorem 3.17. We leave the details to the interested reader. \square

Remark 3.20. The convergence rates proved in Theorem 3.15–3.19 have been observed in numerical experiments. We refer the reader to Chapter 6 for details.

3.6. Epilogue

We like to comment that the analysis of Section 3.1–3.5 can be easily extended to the cases of the other two boundary conditions in (2.11). We note that in the case (2.11)₂ boundary condition (3.9) should be replaced by

$$u_{rrr}^\varepsilon(R) + \frac{n-1}{R}u_{rr}^\varepsilon(R) = \varepsilon,$$

and (3.13) should be replaced by

$$w_{rr}^\varepsilon(R) - \frac{n-1}{R}w_r^\varepsilon(R) = R^{n-1}\varepsilon.$$

We also reiterate an interesting property of the vanishing moment method which was briefly touched on at the end of Chapter 2. That is, the ability of the vanishing moment method to approximate the *concave* solution of the Monge-Ampère problem (1.11)–(1.12). This can be achieved simply by letting $\varepsilon \nearrow 0^-$ in (2.9)–(2.11)₁. This property can be easily proved as follows in the radially symmetric case.

Before giving the proof, we note that for a given $f > 0$ in Ω , equation (1.11) does not have a concave solution in odd dimensions (i.e., n is odd) because $\det(D^2u) = f$ does not hold for any concave function u as all n eigenvalues of Hessian D^2u of a concave function u must be nonpositive. On the other hand, in even dimensions (i.e., n is even), it is trivial to check that if u is a convex solution of problem (1.11)–(1.12) with $g = 0$, then $-u$, which is a concave function, must also be a solution of problem (1.11)–(1.12).

Next, by the same token, it is easy to prove that if u^ε is a convex or “almost convex” solution to problem (2.9)–(2.11)₁, then $-u^\varepsilon$, which is concave or “almost concave”⁵, must also be a solution of (2.9)–(2.11)₁.

Finally, let n be a positive even integer, it is easy to see that changing u^ε to $-u^\varepsilon$ in (2.9)–(2.11)₁ is equivalent to changing ε to $-\varepsilon$ in (2.9)–(2.11)₁. For $\varepsilon < 0$, let $\delta := -\varepsilon$. After replacing ε by $-\delta$ and u^ε by $\hat{u}^\delta := -u^\varepsilon$ in (3.6)–(3.9), we see that \hat{u}^δ satisfies the same set of equations (3.6)–(3.9) with $\delta(> 0)$ in place of ε . Hence, by the analysis of Section 3.2–3.5 we know that there exists a monotone increasing solution \hat{u}^δ to problem (3.6)–(3.9) with ε being replaced by δ , which satisfies all the properties proved in Section 3.2–3.5. Translating all these to $u^\varepsilon = -\hat{u}^\delta$ we conclude that problem (3.6)–(3.9) for $\varepsilon < 0$ has a monotone decreasing solution which is either concave or “almost concave” in $(0, R)$ and converges to the unique concave solution of problem (1.11)–(1.12) as $\varepsilon \nearrow 0^-$. In addition, u^ε satisfies the error estimates stated in Theorem 3.15 and 3.17.

The final comment we like to make is about the possible but well-behaved boundary layer generated by the vanishing moment solution u^ε . In the worst case scenario, the boundary layer, where u^ε may cease to be convex, is confined in an $O(\varepsilon)$ -neighborhood of the boundary $\partial\Omega$. This nice behavior of the boundary layer can be exploited in numerical computations. Indeed, in Chapter 7 we propose an iterative surgical procedure to take advantage of this property of the (possible) boundary layer. We refer the reader to Chapter 7 for the detailed description of the procedure and numerical experiments which show the effectiveness of the proposed iterative surgical procedure.

⁵A function φ^ε is said to be “almost concave” in Ω if it is concave in Ω minus an $O(\varepsilon)$ -neighborhood of the boundary $\partial\Omega$ of Ω .

CHAPTER 4

Conforming finite element approximations

The goal of this chapter is to construct and analyze C^1 finite element approximations for the general fully nonlinear second order Dirichlet problem (2.7)–(2.8) based upon the vanishing moment methodology introduced in Chapter 2 and further analyzed in Chapter 3. Letting u^ε be the solution to problem (2.9)–(2.11)₁, we construct and analyze conforming finite element methods to approximate u^ε using a class of C^1 finite elements such as Argyris, Bell, Bogner-Fox-Schmit, and Hsieh-Clough-Tocher elements (cf. [22]). As a result, we obtain convergent numerical methods for fully nonlinear second order PDEs.

We note that finite element approximations of fourth order PDEs, in particular, the biharmonic equation, were carried out extensively in the seventies for the two-dimensional case [22], and have attracted renewed interests lately for generalizing the well-known two-dimensional finite elements to the three-dimensional case (cf. [73, 77, 78]). Although all of these methods can be readily adapted to discretize problem (2.9)–(2.11)₁, the convergence analysis does not come easy due to the strong nonlinearity of the PDE (2.9). For example, to use the standard perturbation technique for deriving error estimates (a technique successfully used for linear and mildly nonlinear problems), we would have to assume very stringent conditions on the nonlinear differential operator F , which would rule out many interesting application problems, and hence, should be avoided. Instead, we assume very mild conditions on the operator (see Section 4.1 for details), and use a combined fixed-point and linearization technique to simultaneously prove existence and uniqueness for the numerical solution, and also derive error estimates.

The remainder of the chapter is organized as follows. First in Section 4.1, we give additional notation, and then define the finite element method based upon the variational formulation (2.12). Next, we make certain structure assumptions about the nonlinear differential operator F which will play an important role in our analysis. In Section 4.2, we show existence of solutions of the linearized problem and prove stability and convergence results of its finite element approximations. The main results of the chapter are found in Section 4.3, where we use a fixed point argument to simultaneously show existence, uniqueness, and convergence of the finite element approximation of (2.9)–(2.11)₁.

4.1. Formulation of conforming finite element methods

First, we introduce the following function space notation:

$$V := H^2(\Omega), \quad V_0 := H^2(\Omega) \cap H_0^1(\Omega), \quad V_g := \{v \in V; v|_{\partial\Omega} = g\}.$$

Let \mathcal{T}_h be a quasiuniform triangular or rectangular partition of Ω , and let $V^h \subset V$ be a conforming finite element space consisting of piecewise polynomials

of degree $k > 4$ such that for any $v \in V \cap H^s(\Omega)$, we have

$$(4.1) \quad \inf_{v_h \in V^h} \|v - v_h\|_{H^j} \leq Ch^{\ell-j} \|v\|_{H^\ell} \quad j = 0, 1, 2, \quad \ell = \min\{s, k+1\}.$$

Let

$$(4.2) \quad V_0^h := \{v_h \in V^h; v_h|_{\partial\Omega} = 0\}, \quad V_g^h := \{v_h \in V^h; v_h|_{\partial\Omega} = g\}.$$

Based on (2.12), we define the finite element formulation of (2.9)–(2.11) as to find $u_h^\varepsilon \in V_g^h$ such that

$$(4.3) \quad \varepsilon(\Delta u_h^\varepsilon, \Delta v_h) + (F(D^2 u_h^\varepsilon, \nabla u_h^\varepsilon, u_h^\varepsilon, x), v_h) = \left\langle \varepsilon^2, \frac{\partial v_h}{\partial \nu} \right\rangle_{\partial\Omega} \quad \forall v_h \in V_0^h.$$

Let u^ε be the solution to (2.12) and let u_h^ε be a solution to (4.3). The primary goal of this chapter is to derive error estimates of $u^\varepsilon - u_h^\varepsilon$, which then means we need to first prove that there exists $u_h^\varepsilon \in V_g^h$ solving (4.3), and that u_h^ε is unique. Clearly, we must assume some structure conditions on the nonlinear differential operator F to achieve any of these goals. Indeed, the assumptions that we make will play an important role in our results and in the techniques to derive them. We refer to Section 1.2 for the notation used in this chapter.

Assumption (A)

[A1] There exists $\varepsilon_0 \in (0, 1)$ such that for all $\varepsilon \in (0, \varepsilon_0]$, there exists a locally unique solution to (2.9)–(2.11)₁ with $u^\varepsilon \in H^s(\Omega)$ ($s \geq 3$).

[A2] For $\varepsilon \in (0, \varepsilon_0]$, the operator $(G'_\varepsilon[u^\varepsilon])^*$ (the adjoint of $G'_\varepsilon[u^\varepsilon]$) is an isomorphism from V_0 to V_0^* . That is for all $\varphi \in V_0^*$ (the dual space of V_0), there exists $v \in V_0$ such that

$$(4.4) \quad \left\langle (G'_\varepsilon[u^\varepsilon])^*(v), w \right\rangle = \langle \varphi, w \rangle \quad \forall w \in V_0.$$

Here, $\langle \cdot, \cdot \rangle$ denotes the dual pairing between V_0 and V_0^* . Furthermore, there exists positive constants $C_0 = C_0(\varepsilon)$, $C_1 = C_1(\varepsilon)$ such that the following Gårding inequality holds:

$$(4.5) \quad \langle G'_\varepsilon[u^\varepsilon](v), v \rangle \geq C_1 \|v\|_{H^2}^2 - C_0 \|v\|_{L^2}^2 \quad \forall v \in V_0,$$

and there exists $C_2 = C_2(\varepsilon) > 0$ such that

$$\|F'[u^\varepsilon]\|_{VV^*} \leq C_2,$$

where

$$\|F'[u^\varepsilon]\|_{VV^*} := \sup_{v \in V_0} \frac{\|F'[u^\varepsilon](v)\|_{H^{-2}}}{\|v\|_{H^2}} := \sup_{v \in V_0} \sup_{w \in V_0} \frac{\langle F'[u^\varepsilon](v), w \rangle}{\|v\|_{H^2} \|w\|_{H^2}}.$$

Moreover, there exists $p > 2$ and $C_R = C_R(\varepsilon) > 0$ such that if $\varphi \in L^2(\Omega)$ and $v \in V_0$ satisfies (4.4), then $v \in H^p(\Omega)$ and

$$\|v\|_{H^p} \leq C_R \|\varphi\|_{L^2}.$$

[A3] There exists a Banach space Y with norm $\|\cdot\|_Y$ that is well-defined and finite on V^h , and a constant $C > 0$, independent of ε , such that

$$\sup_{y \in Y} \frac{\|F'[y]\|_{VV^*}}{\|y\|_Y} \leq C.$$

[A4] There exists a constant $C > 0$ independent of ε such that

$$\|\mathcal{I}^h u^\varepsilon\|_Y \leq C \|u^\varepsilon\|_Y,$$

where $\mathcal{I}^h u^\varepsilon \in V_g^h$ denotes the finite element interpolant of u^ε .

[A5] There exists a constant $\delta = \delta(\varepsilon) \in (0, 1)$, such that for any $w_h \in V_g^h$ with $\|\mathcal{I}^h u^\varepsilon - w_h\|_{H^2} \leq \delta$, there holds

$$\|F'[u^\varepsilon] - F'[w_h]\|_{VV^*} \leq L(h) \|u^\varepsilon - w_h\|_{H^2},$$

where $L(h) = L(\varepsilon, h)$ may depend on both h and ε and satisfies $L(h) = o(h^{2-\ell})$.

Remark 4.1. (a) Conditions [A1]–[A5] are fairly mild, and a very large class of fully nonlinear second order differential operators satisfy these requirements (cf. Chapter 6). Clearly, we must assume [A1] in order for the finite element method (4.3) to have any significance, and the regularity requirements of u^ε are needed to obtain any meaningful error estimates.

(b) Condition [A2] is naturally satisfied if $-F$ is elliptic at u^ε (cf. [42, Chapter 17]), and the regularity requirements are expected to hold provided that u^ε and $\partial\Omega$ are sufficiently regular.

(c) By standard interpolation theory [22, 13], there holds

$$(4.6) \quad \|u^\varepsilon - \mathcal{I}^h u^\varepsilon\|_{H^j} \leq Ch^{\ell-j} \|u^\varepsilon\|_{H^\ell} \quad j = 0, 1, 2, \quad \ell = \min\{s, k+1\}.$$

(d) Condition [A5], which states that F' is locally Lipschitz near u^ε , is the strongest requirement among the five listed, and it is the authors' experience that this is the most difficult property to verify. As one may expect, this assumption plays an important role in the fixed point argument, which is needed in our analysis to obtain existence, uniqueness, and error estimates of the finite element method (4.3).

4.2. Linearization and its finite element approximations

To construct the necessary tools to analyze the finite element method (4.3), we first study finite element approximation of the linearization of (2.9). We note that the materials of this section have an independent interest within themselves. To the best of our knowledge, finite element error estimates for non-coercive linear fourth order problems have not been explicitly given in the literature before.

4.2.1. Linearization. For given $\varphi \in V_0^*$ and $\psi \in H^{-\frac{1}{2}}(\partial\Omega)$, we consider the following linear problem:

$$(4.7) \quad G'_\varepsilon[u^\varepsilon](v) = \varphi \quad \text{in } \Omega,$$

$$(4.8) \quad v = 0 \quad \text{on } \partial\Omega,$$

$$(4.9) \quad \Delta v = \psi \quad \text{on } \partial\Omega.$$

Multiplying the equation (4.7) by $w \in V_0$, integrating over Ω , and integrating by parts, we obtain

$$\langle G'_\varepsilon[u^\varepsilon](v), w \rangle = \varepsilon(\Delta v, \Delta w) + \langle F'[u^\varepsilon](v), w \rangle - \varepsilon \left\langle \Delta v, \frac{\partial w}{\partial \nu} \right\rangle_{\partial\Omega}.$$

Based on this calculation, we define the weak formulation of (4.7)–(4.9) as to find $v \in V_0$ such that

$$(4.10) \quad a^\varepsilon(v, w) = \langle \varphi, w \rangle + \varepsilon \left\langle \psi, \frac{\partial w}{\partial \nu} \right\rangle_{\partial \Omega} \quad \forall w \in V_0,$$

where

$$a^\varepsilon(v, w) := \varepsilon(\Delta v, \Delta w) + \langle F'[u^\varepsilon](v), w \rangle.$$

In view of assumptions [A1]–[A2], we immediately have the following theorem.

Theorem 4.2. *Suppose assumptions [A1]–[A2] hold. Then there exists a unique solution $v \in V_0$ to (4.10). Furthermore, there exists $C_3 = C_3(\varepsilon) > 0$ such that*

$$(4.11) \quad \|v\|_{H^2} \leq C_3 \left(\|\varphi\|_{H^{-2}} + \varepsilon \|\psi\|_{H^{-\frac{1}{2}}(\partial \Omega)} \right).$$

PROOF. From the Gårding inequality (4.5) and the fact $(G'_\varepsilon[u^\varepsilon])^*$ is injective on V_0 , it follows that $G'_\varepsilon[u^\varepsilon]$ is an isomorphism from V_0 to V_0^* using a Fredholm alternative argument [1, Theorem 8.5].

We now claim that there exists $C_S = C_S(\varepsilon)$ such that $\|v\|_{L^2} \leq C_S(\|\varphi\|_{H^{-2}} + \varepsilon \|\psi\|_{H^{-\frac{1}{2}}(\partial \Omega)})$. If not, there would exist sequences $\{\varphi_m\}_{m=1}^\infty \subset V_0^*$, $\{\psi_m\}_{m=1}^\infty \subset H^{-\frac{1}{2}}(\partial \Omega)$, and $\{v_m\}_{m=1}^\infty \subset V_0$ such that

$$\langle G'_\varepsilon[u^\varepsilon](v_m), w \rangle = \langle \varphi_m, w \rangle + \varepsilon \left\langle \psi_m, \frac{\partial w}{\partial \nu} \right\rangle_{\partial \Omega} \quad w \in V_0,$$

but

$$\|v_m\|_{L^2} > m(\|\varphi_m\|_{H^{-2}} + \varepsilon \|\psi_m\|_{H^{-\frac{1}{2}}(\partial \Omega)}).$$

Without loss of generality, we may as well suppose $\|v_m\|_{L^2} = 1$ (and therefore $\|\varphi_m\|_{H^{-2}} + \varepsilon \|\psi_m\|_{H^{-\frac{1}{2}}(\partial \Omega)} \rightarrow 0$ as $m \rightarrow \infty$). In light of (4.5), $\{v_m\}_{m=1}^\infty$ is bounded in V_0 , and hence by a compactness argument, there exists a subsequence $\{v_{m_j}\}_{m=1}^\infty$ and $v \in V_0$ such that

$$(4.12) \quad v_{m_j} \rightharpoonup v \quad \text{weakly in } V_0,$$

$$(4.13) \quad v_{m_j} \rightarrow v \quad \text{in } H_0^1(\Omega).$$

Therefore,

$$\langle G'_\varepsilon[u^\varepsilon](v), w \rangle = 0 \quad \forall w \in V_0.$$

Since $G'_\varepsilon[u^\varepsilon]$ is an isomorphism, $v \equiv 0$. However (4.13) implies that $\|v\|_{L^2} = 1$, a contradiction.

Hence there exists C_S such that

$$\|v\|_{L^2} \leq C_S(\|\varphi\|_{H^{-2}} + \varepsilon \|\psi\|_{H^{-\frac{1}{2}}(\partial \Omega)}),$$

and therefore by (4.5) and a trace inequality, we have

$$\begin{aligned}
C_1 \|v\|_{H^2}^2 &\leq \varepsilon(\Delta v, \Delta v) + \langle F'[u^\varepsilon](v), v \rangle + C_0 \|v\|_{L^2}^2 \\
&= a^\varepsilon(v, v) + C_0 \|v\|_{L^2}^2 \\
&= \langle \varphi, v \rangle + \varepsilon \left\langle \psi, \frac{\partial v}{\partial \nu} \right\rangle_{\partial \Omega} + C_0 \|v\|_{L^2}^2 \\
&\leq C \left(\|\varphi\|_{H^{-2}} + \varepsilon \|\psi\|_{H^{-\frac{1}{2}}(\partial \Omega)} + C_0 \|v\|_{L^2} \right) \|v\|_{H^2} \\
&\leq C(1 + C_0 C_S) \left(\|\varphi\|_{H^{-2}} + \varepsilon \|\psi\|_{H^{-\frac{1}{2}}(\partial \Omega)} \right) \|v\|_{H^2}.
\end{aligned}$$

Dividing by $C_1 \|v\|_{H^2}$, we obtain (4.11) with $C_3 = CC_1^{-1}(1 + C_0 C_S)$. \square

4.2.2. Finite element approximation. Let $V_0^h \subset V_0$ be one of the finite dimensional subspaces of degree $k > 4$ defined in Section 4.1. Based on the variational formulation (4.10), we define the finite element method for (4.7)–(4.9) as to find $v_h \in V_0^h$ such that

$$(4.14) \quad a^\varepsilon(v_h, w_h) = \langle \varphi, w_h \rangle + \varepsilon \left\langle \psi, \frac{\partial w_h}{\partial \nu} \right\rangle_{\partial \Omega} \quad \forall w_h \in V_0^h.$$

Using a modification of the well-known Schatz's argument (cf. [13, Theorem 5.7.6]), we obtain the following result.

Theorem 4.3. *Let assumptions [A1]–[A2] hold and suppose that $v \in H^s(\Omega)$ ($s \geq 3$) is the unique solution to (4.10). Then for $h \leq h_0(\varepsilon)$, there exists a unique solution $v_h \in V_0^h$ to (4.14), where*

$$(4.15) \quad h_0 = \begin{cases} C (C_0 C_1^{-1} C_2^2 C_R^2)^{\frac{1}{4-2r}} & \text{if } C_0 \neq 0, \\ 1 & \text{if } C_0 = 0, \end{cases} \quad r = \min\{p, k+1\}.$$

Furthermore, there holds the following inequalities:

$$(4.16) \quad \|v - v_h\|_{H^2} \leq C_4 h^{\ell-2} \|v\|_{H^\ell},$$

$$(4.17) \quad \|v - v_h\|_{L^2} \leq C_5 h^{\ell+r-4} \|v\|_{H^\ell},$$

where

$$C_4 = C_4(\varepsilon) = CC_1^{-1}C_2, \quad C_5 = C_5(\varepsilon) = CC_1^{-1}C_2^2C_R, \quad \ell = \min\{s, k+1\}.$$

PROOF. To show existence, we begin by deriving estimates for a solution v_h to (4.14) that may exist. We start with the error equation:

$$a^\varepsilon(v - v_h, w_h) = 0 \quad \forall w_h \in V_0^h.$$

Then using (4.5) and [A2], we have for any $w_h \in V_0^h$

$$\begin{aligned}
C_1 \|v - v_h\|_{H^2}^2 &= a^\varepsilon(v - v_h, v - v_h) + C_0 \|v - v_h\|_{L^2}^2 \\
&= a^\varepsilon(v - v_h, v - w_h) + C_0 \|v - v_h\|_{L^2}^2 \\
&\leq \varepsilon \|\Delta(v - v_h)\|_{L^2} \|\Delta(v - w_h)\|_{L^2} \\
&\quad + \|F'[u^\varepsilon]\|_{V^*} \|v - v_h\|_{H^2} \|v - w_h\|_{H^2} + C_0 \|v - v_h\|_{L^2}^2 \\
&\leq CC_2 \|v - v_h\|_{H^2} \|v - w_h\|_{H^2} + C_0 \|v - v_h\|_{L^2}^2.
\end{aligned}$$

Thus, by (4.1)

$$(4.18) \quad C_1 \|v - v_h\|_{H^2}^2 \leq CC_1^{-1} C_2^2 h^{2\ell-4} \|v\|_{H^\ell}^2 + C_0 \|v - v_h\|_{L^2}^2.$$

Next, we let $w \in V_0 \cap H^p(\Omega)$ ($p > 2$) be the solution to the following auxiliary problem:

$$\langle (G'_\varepsilon[u^\varepsilon])^*(w), z \rangle = (v - v_h, z) \quad \forall z \in V_0.$$

By assumption [A2], there exists such a solution w with

$$(4.19) \quad \|w\|_{H^p} \leq C_R \|v - v_h\|_{L^2}.$$

We then have for any $w_h \in V_0^h$

$$\begin{aligned} \|v - v_h\|_{L^2}^2 &= \langle (G'_\varepsilon[u^\varepsilon])^*(w), (v - v_h) \rangle \\ &= \langle G'_\varepsilon[u^\varepsilon](v - v_h), w \rangle \\ &= a^\varepsilon(v - v_h, w) \\ &= a^\varepsilon(v - v_h, w - w_h) \\ &\leq CC_2 \|v - v_h\|_{H^2} \|w - w_h\|_{H^2}. \end{aligned}$$

Consequently from (4.1) and (4.19)

$$\begin{aligned} \|v - v_h\|_{L^2}^2 &\leq CC_2 h^{r-2} \|v - v_h\|_{H^2} \|w\|_{H^p} \\ &\leq CC_2 C_R h^{r-2} \|v - v_h\|_{H^2} \|v - v_h\|_{L^2}, \end{aligned}$$

and thus,

$$(4.20) \quad \|v - v_h\|_{L^2} \leq CC_2 C_R h^{r-2} \|v - v_h\|_{H^2}.$$

Applying the inequality (4.20) into (4.18) gives us

$$\begin{aligned} C_1 \|v - v_h\|_{H^2}^2 &\leq CC_1^{-1} C_2^2 h^{2\ell-4} \|v\|_{H^\ell}^2 + C_0 \|v - v_h\|_{L^2}^2 \\ &\leq CC_1^{-1} C_2^2 h^{2\ell-4} \|v\|_{H^\ell}^2 + CC_0 C_2^2 C_R^2 h^{2r-4} \|v - v_h\|_{H^2}^2. \end{aligned}$$

Thus, for $h \leq h_0$

$$C_1 \|v - v_h\|_{H^2}^2 \leq CC_1^{-1} C_2^2 h^{2\ell-4} \|v\|_{H^\ell}^2,$$

and therefore

$$\begin{aligned} \|v - v_h\|_{H^2} &\leq CC_1^{-1} C_2 h^{\ell-2} \|v\|_{H^\ell}, \\ \|v - v_h\|_{L^2} &\leq CC_1^{-1} C_2^2 C_R h^{\ell+r-4} \|v\|_{H^\ell}. \end{aligned}$$

So far, we have been under the assumption that there exists a solution v_h . We now consider the question of existence and uniqueness. First, since the problem under consideration is linear and in a finite dimensional setting, existence and uniqueness are equivalent. Now suppose $\varphi \equiv 0$, $\psi \equiv 0$. In light of (4.11), we have $v \equiv 0$, and therefore, (4.16) implies $v_h \equiv 0$ as well provided that h is sufficiently small. In particular, this means that (4.14) has a unique solution for $h \leq h_0$. \square

Remark 4.4. (a) Because (4.4) is a fourth order problem, we expect $p \geq 3$. Therefore, since the polynomial degree k is strictly greater than four, we expect $r = p$ in Theorem 4.3.

(b) In many cases, it is possible to get a relatively good idea of how the constant C_R depends on ε . To see this, suppose that there exists a constant $\widehat{C}_2 > 0$ such that if v solves (4.4), then $(F'[u^\varepsilon](v))^* \in H^{-1}(\Omega)$ and

$$\|(F'[u^\varepsilon])^*(v)\|_{H^{-1}} \leq \widehat{C}_2 \|v\|_{H^1}.$$

Here, $(F'[u^\varepsilon])^*$ denotes the adjoint operator of $F'[u^\varepsilon]$.

Now if $p = 3$ in [A2], then

$$\langle (G'_\varepsilon[u^\varepsilon])^*(v), \Delta v \rangle = (\varphi, \Delta v),$$

where $\langle \cdot, \cdot \rangle$ now denotes the dual pairing of $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Therefore, after integrating by parts

$$\begin{aligned} \varepsilon \|\nabla \Delta v\|_{L^2}^2 &= \langle (F'[u^\varepsilon])^*(v), \Delta v \rangle - (\varphi, \Delta v) \\ &\leq (\widehat{C}_2 \|v\|_{H^1} + \|\varphi\|_{H^{-1}}) \|\Delta v\|_{H^1}. \end{aligned}$$

Hence, by Poincare's inequality

$$(4.21) \quad \|\nabla \Delta v\|_{L^2} \leq C\varepsilon^{-1} (\widehat{C}_2 \|v\|_{H^1} + \|\varphi\|_{L^2}).$$

By the proof of Theorem 4.2, it is apparent that

$$\|v\|_{H^2} \leq CC_3 \|\varphi\|_{L^2},$$

and therefore

$$(4.22) \quad \|\nabla \Delta v\|_{L^2} \leq C\varepsilon^{-1} (\widehat{C}_2 C_3 + 1) \|\varphi\|_{L^2} \leq C\varepsilon^{-1} \widehat{C}_2 C_3 \|\varphi\|_{L^2}.$$

Furthermore, if $(F'[u^\varepsilon])^*$ is coercive on $H_0^1(\Omega)$, that is, there exists a constant $\widehat{C}_1 > 0$ such that

$$(4.23) \quad \langle (F'[u^\varepsilon])^*(v), v \rangle \geq \widehat{C}_1 \|v\|_{H^1}^2 \quad \forall v \in H_0^1(\Omega),$$

then by (4.21)

$$(4.24) \quad \|\nabla \Delta v\|_{L^2} \leq C\varepsilon^{-1} \widehat{C}_1^{-1} \widehat{C}_2 \|\varphi\|_{L^2}.$$

In view of (4.22) or (4.24), we can expect that in the general case

$$\|v\|_{H^3} \leq C\varepsilon^{-1} \widehat{C}_2 C_3 \|\varphi\|_{L^2},$$

and if (4.23) holds

$$\|v\|_{H^3} \leq C\varepsilon^{-1} \widehat{C}_1^{-1} \widehat{C}_2 \|\varphi\|_{L^2}.$$

Hence, for $p = 3$ we have $C_R = C\varepsilon^{-1} \widehat{C}_2 C_3$ in the general case and $C_R = C\varepsilon^{-1} \widehat{C}_1^{-1} \widehat{C}_2$ if (4.23) holds.

Now we consider the case $p = 4$, and for simplicity, we assume $F'[u^\varepsilon]$ is self-adjoint. We then have

$$\langle G'_\varepsilon[u^\varepsilon](v), \Delta^2 v \rangle = (\varphi, \Delta^2 v),$$

and therefore

$$\begin{aligned}\varepsilon \|\Delta^2 v\|_{L^2}^2 &= (\varphi, \Delta^2 v) - \langle F'[u^\varepsilon](v), \Delta^2 v \rangle \\ &\leq C \left(\|\varphi\|_{L^2} + \|F'[u^\varepsilon]\|_\infty \|v\|_{H^2} \right) \|\Delta^2 v\|_{L^2} \\ &\leq C \left(1 + C_3 \|F'[u^\varepsilon]\|_\infty \right) \|\varphi\|_{L^2} \|\Delta^2 v\|_{L^2},\end{aligned}$$

where we define

$$\|F'[u^\varepsilon]\|_\infty := \max_{1 \leq i, j \leq n} \left\| \frac{\partial F(u^\varepsilon)}{\partial r_{ij}} \right\|_{L^\infty} + \max_{1 \leq i \leq n} \left\| \frac{\partial F(u^\varepsilon)}{\partial p_i} \right\|_{L^\infty} + \left\| \frac{\partial F(u^\varepsilon)}{\partial z} \right\|_{L^\infty}.$$

We then expect that in this case that

$$\|v\|_{H^4} \leq CC_3 \varepsilon^{-1} \|F'[u^\varepsilon]\|_\infty \|\varphi\|_{L^2}.$$

Hence, for $p = 4$ we have $C_R = CC_3 \varepsilon^{-1} \|F'[u^\varepsilon]\|_\infty$.

4.3. Convergence analysis of finite element approximation

In this section, we give the main results of this chapter, where we establish existence and uniqueness, and derive error estimates for the finite element method (4.3). First, we define an operator $T_h : V_g^h \mapsto V_g^h$ such that for a given $v_h \in V_g^h$, $T_h(v_h)$ is the solution to the following linear problem:

$$\begin{aligned}(4.25) \quad a^\varepsilon(v_h - T_h(v_h), w_h) \\ = \varepsilon(\Delta v_h, \Delta w_h) + \langle F(v_h), w_h \rangle - \left\langle \varepsilon^2, \frac{\partial w_h}{\partial \nu} \right\rangle_{\partial \Omega} \quad \forall w_h \in V_0^h.\end{aligned}$$

In view of Theorem 4.3, T_h is well-defined provided that assumptions [A1]–[A2] hold and $h \leq h_0$. We note that the right-hand side of (4.25) is the residual of the finite element method (4.3), and therefore, any fixed point of T_h (i.e. $T(v_h) = v_h$) is a solution to (4.3) and vice-versa. Our goal is to show that indeed, T_h has a unique fixed point in a small neighborhood of u^ε . To this end, we define the following ball:

$$\mathbb{B}_h(\rho) := \{v_h \in V_g^h; \|\mathcal{I}^h u^\varepsilon - v_h\|_{H^2} \leq \rho\},$$

where the center of the ball $\mathcal{I}^h u^\varepsilon$ is the finite element interpolant of u^ε .

For the continuation of this chapter, we let $\ell = \min\{s, k + 1\}$, where we recall that k is the polynomial degree of the finite element space V^h and s is defined in [A1]. The following lemma shows that the distance between the center of \mathbb{B}_h and its image under T_h is small.

Lemma 4.5. *Suppose assumptions [A1]–[A4] hold. Then for $h \leq h_0(\varepsilon)$,*

$$(4.26) \quad \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{H^2} \leq C_6 h^{\ell-2} \|u^\varepsilon\|_{H^\ell},$$

where

$$C_6 = C_6(\varepsilon) = CC_1^{-\frac{1}{2}} \|u^\varepsilon\|_Y \max\{C_1^{-\frac{1}{2}}, C_0^{\frac{1}{2}} C_R\}.$$

PROOF. To ease notation, set $r^\varepsilon = \mathcal{I}^h u^\varepsilon - u^\varepsilon$. Using the definition of $T_h(\cdot)$ and the mean value theorem, we have for any $z_h \in V_0^h$

$$\begin{aligned}
 (4.27) \quad a^\varepsilon(\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon), z_h) &= \varepsilon(\Delta \mathcal{I}^h u^\varepsilon, \Delta z_h) + (F(\mathcal{I}^h u^\varepsilon), z_h) - \left\langle \varepsilon^2, \frac{\partial z_h}{\partial \nu} \right\rangle_{\partial \Omega} \\
 &= \varepsilon(\Delta r^\varepsilon, \Delta z_h) + (F(\mathcal{I}^h u^\varepsilon) - F(u^\varepsilon), z_h) \\
 &= \varepsilon(\Delta r^\varepsilon, \Delta z_h) + \langle F'[y_h](r^\varepsilon), z_h \rangle,
 \end{aligned}$$

where $y_h = \mathcal{I}^h u^\varepsilon - \gamma r^\varepsilon$ for some $\gamma \in [0, 1]$.

Setting $z_h = \mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)$ and making use of [A2]–[A4], we have

$$\begin{aligned}
 C_1 \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{H^2}^2 &\leq \varepsilon \|r^\varepsilon\|_{H^2} \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{H^2} \\
 &\quad + C \|u^\varepsilon\|_Y \|r^\varepsilon\|_{H^2} \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{H^2} + C_0 \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{L^2}^2,
 \end{aligned}$$

and so by the Cauchy-Schwarz inequality,

$$\begin{aligned}
 (4.28) \quad C_1 \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{H^2}^2 &\leq C_1^{-1} \varepsilon^2 \|r^\varepsilon\|_{H^2}^2 + C C_1^{-1} \|u^\varepsilon\|_Y^2 \|r^\varepsilon\|_{H^2}^2 + C_0 \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{L^2}^2 \\
 &\leq C C_1^{-1} h^{2\ell-4} \|u^\varepsilon\|_Y^2 \|u^\varepsilon\|_{H^\ell}^2 + C_0 \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{L^2}^2.
 \end{aligned}$$

Next, we let $w \in V_0 \cap H^p(\Omega)$ ($p > 2$) be the solution to the following auxiliary problem:

$$\langle (G'_\varepsilon[u^\varepsilon])^*(w), z \rangle = (\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon), z) \quad \forall z \in V_0,$$

with

$$(4.29) \quad \|w\|_{H^p} \leq C_R \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{L^2}.$$

Then for any $z_h \in V_0^h$ we get

$$\begin{aligned}
 &\|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{L^2}^2 \\
 &= a(\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon), w) \\
 &= a(\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon), w - z_h) + \varepsilon(\Delta r^\varepsilon, \Delta z_h) + \langle F'[y_h](r^\varepsilon), z_h \rangle \\
 &\leq C C_2 \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{H^2} \|w - z_h\|_{H^2} + \varepsilon \|\Delta r^\varepsilon\|_{L^2} \|\Delta z_h\|_{L^2} \\
 &\quad + C \|u^\varepsilon\|_Y \|r^\varepsilon\|_{H^2} \|z_h\|_{H^2}.
 \end{aligned}$$

Taking $z_h = \mathcal{I}^h w$, we have from (4.6) and (4.29)

$$\begin{aligned}
 &\|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{L^2}^2 \\
 &\leq C C_R^2 \left(C_2^2 h^{2r-4} \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{H^2}^2 + h^{2\ell-4} \|u^\varepsilon\|_Y^2 \|u^\varepsilon\|_{H^\ell}^2 \right).
 \end{aligned}$$

Substituting this last bound into the inequality (4.28) we have

$$\begin{aligned}
 C_1 \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{H^2}^2 &\leq C \left((C_1^{-1} + C_0 C_R^2) h^{2\ell-4} \|u^\varepsilon\|_Y^2 \|u^\varepsilon\|_{H^\ell}^2 \right. \\
 &\quad \left. + C_0 C_2^2 C_R^2 h^{2r-4} \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{H^2}^2 \right).
 \end{aligned}$$

It then follows that for $h \leq h_0$

$$\|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{H^2} \leq C C_1^{-\frac{1}{2}} (C_1^{-\frac{1}{2}} + C_0^{\frac{1}{2}} C_R) h^{\ell-2} \|u^\varepsilon\|_Y \|u^\varepsilon\|_{H^\ell},$$

which is the inequality (4.26). The proof is complete. \square

Lemma 4.6. *Suppose assumptions [A1]–[A5] hold. Then there exists an $h_1 = h_1(\varepsilon) > 0$ such that for $h \leq \min\{h_0, h_1\}$, the operator T_h is a contracting mapping in the ball $\mathbb{B}_h(\rho_0)$ with a contraction factor $\frac{1}{2}$, that is*

$$\|T_h(v_h) - T_h(w_h)\|_{H^2} \leq \frac{1}{2} \|v_h - w_h\|_{H^2} \quad \forall v_h, w_h \in \mathbb{B}_h(\rho_0),$$

where

$$\rho_0 = \min \left\{ \delta, CC_1^{\frac{1}{2}} L^{-1}(h) \min\{C_1^{\frac{1}{2}}, C_0^{-\frac{1}{2}} C_R^{-1}\} \right\},$$

and h_1 is chosen such that

$$h_1 = C \left(C_1^{-\frac{1}{2}} L(h_1) \max\{C_1^{-\frac{1}{2}}, C_0^{\frac{1}{2}} C_R\} \right)^{\frac{1}{2-\ell}}.$$

PROOF. By the definition of T_h , we have for any $v_h, w_h \in \mathbb{B}_h(\rho_0)$, $z_h \in V_0^h$,

$$\begin{aligned} a^\varepsilon(T_h(v_h) - T_h(w_h), z_h) &= a^\varepsilon(v_h, z_h) - a^\varepsilon(w_h, z_h) + \varepsilon(\Delta(w_h - v_h), \Delta z_h) \\ &\quad + (F(w_h) - F(v_h), z_h) \\ &= \langle F'[u^\varepsilon](v_h - w_h), z_h \rangle + (F(w_h) - F(v_h), z_h). \end{aligned}$$

Using the mean value theorem, we obtain

$$\begin{aligned} a^\varepsilon(T_h(v_h) - T_h(w_h), z_h) &= \langle F'[u^\varepsilon](v_h - w_h), z_h \rangle + (F(w_h) - F(v_h), z_h) \\ &= \langle (F'[u^\varepsilon] - F'[y_h])(v_h - w_h), z_h \rangle, \end{aligned}$$

where $y_h = w_h + \gamma(v_h - w_h)$ for some $\gamma \in [0, 1]$. Here, we have abused the notation of y_h , defining it differently in two different proofs in this section.

Using [A2], [A5], and the triangle inequality yields

$$\begin{aligned} &C_1 \|T_h(v_h) - T_h(w_h)\|_{H^2}^2 \\ &\leq \|F'[u^\varepsilon] - F'[y_h]\|_{VV^*} \|v_h - w_h\|_{H^2} \|T_h(v_h) - T_h(w_h)\|_{H^2} \\ &\quad + C_0 \|T_h(v_h) - T_h(w_h)\|_{L^2}^2 \\ &\leq L(h) \|u^\varepsilon - y_h\|_{H^2} \|v_h - w_h\|_{H^2} \|T_h(v_h) - T_h(w_h)\|_{H^2} \\ &\quad + C_0 \|T_h(v_h) - T_h(w_h)\|_{L^2}^2 \\ &\leq CL(h) (h^{\ell-2} \|u^\varepsilon\|_{H^\ell} + \rho_0) \|v_h - w_h\|_{H^2} \|T_h(v_h) - T_h(w_h)\|_{H^2} \\ &\quad + C_0 \|T_h(v_h) - T_h(w_h)\|_{L^2}^2. \end{aligned}$$

Thus,

$$(4.30) \quad \begin{aligned} C_1 \|T_h(v_h) - T_h(w_h)\|_{H^2}^2 &\leq C_0 \|T_h(v_h) - T_h(w_h)\|_{L^2}^2 \\ &\quad + CC_1^{-1} L^2(h) (h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 + \rho_0^2) \|v_h - w_h\|_{H^2}^2. \end{aligned}$$

Next, employing a duality argument similar to the one used in Lemma 4.5, we let $w \in V_0 \cap H^p(\Omega)$ ($p > 2$) satisfy

$$\langle (G'_\varepsilon[u^\varepsilon])^*(w), z \rangle = (T_h(v_h) - T_h(w_h), z) \quad \forall z \in V_0,$$

with

$$(4.31) \quad \|w\|_{H^p} \leq C_R \|T_h(v_h) - T_h(w_h)\|_{L^2}.$$

Then using the same methods as in Lemma 4.5, we conclude

$$\begin{aligned} \|T_h(v_h) - T_h(w_h)\|_{L^2}^2 &\leq C \left(L(h)(h^{\ell-2}\|u^\varepsilon\|_{H^\ell} + \rho_0) \|v_h - w_h\|_{H^2} \right. \\ &\quad \left. + C_2 h^{r-2} \|T_h(v_h) - T_h(w_h)\|_{H^2} \right) \|w\|_{H^p} \\ &\leq CC_R \left(L(h)(h^{\ell-2}\|u^\varepsilon\|_{H^\ell} + \rho_0) \|v_h - w_h\|_{H^2} \right. \\ &\quad \left. + C_2 h^{r-2} \|T_h(v_h) - T_h(w_h)\|_{H^2} \right) \|T_h(v_h) - T_h(w_h)\|_{L^2}, \end{aligned}$$

and therefore

$$\begin{aligned} \|T_h(v_h) - T_h(w_h)\|_{L^2}^2 &\leq C \left(C_R^2 L^2(h)(h^{2\ell-4}\|u^\varepsilon\|_{H^\ell}^2 + \rho_0^2) \|v_h - w_h\|_{H^2}^2 \right. \\ &\quad \left. + C_2^2 C_R^2 h^{2r-4} \|T_h(v_h) - T_h(w_h)\|_{H^2}^2 \right). \end{aligned}$$

Using this last inequality in (4.30) gives us

$$\begin{aligned} &C_1 \|T_h(v_h) - T_h(w_h)\|_{H^2}^2 \\ &\leq C \left(L^2(h)(C_1^{-1} + C_0 C_R^2)(h^{2\ell-4}\|u^\varepsilon\|_{H^\ell}^2 + \rho_0^2) \|v_h - w_h\|_{H^2}^2 \right. \\ &\quad \left. + C_0 C_2^2 C_R^2 h^{2r-4} \|T_h(v_h) - T_h(w_h)\|_{H^2}^2 \right). \end{aligned}$$

Therefore, for $h \leq h_0$

$$\begin{aligned} &\|T_h(v_h) - T_h(w_h)\|_{H^2} \\ &\leq CC_1^{-\frac{1}{2}} L(h)(C_1^{-\frac{1}{2}} + C_0^{\frac{1}{2}} C_R)(h^{\ell-2}\|u^\varepsilon\|_{H^\ell} + \rho_0) \|v_h - w_h\|_{H^2}. \end{aligned}$$

It then follows from the definition of ρ_0 and h_1 that for $h \leq \min\{h_0, h_1\}$,

$$\|T_h(v_h) - T_h(w_h)\|_{H^2} \leq \frac{1}{2} \|v_h - w_h\|_{H^2}.$$

□

With these two lemmas in hand, we can now derive the main results of this chapter.

Theorem 4.7. *Under the same hypotheses of Lemma 4.6, there exists $h_2 = h_2(\varepsilon) > 0$ such that for $h \leq \min\{h_0, h_2\}$, there exists a locally unique solution to (4.3), where h_2 is chosen such that*

$$h_2 = C \left(C_6 \|u^\varepsilon\|_{H^\ell} \max \left\{ \delta^{-1}, C_1^{-\frac{1}{2}} L(h_2) \max \{ C_1^{-\frac{1}{2}}, C_0^{\frac{1}{2}} C_R \} \right\} \right)^{\frac{1}{2-\ell}}.$$

Furthermore, there holds the following error estimate:

$$(4.32) \quad \|u^\varepsilon - u_h^\varepsilon\|_{H^2} \leq C_7 h^{\ell-2} \|u^\varepsilon\|_{H^\ell},$$

with

$$C_7 = C_7(\varepsilon) = CC_1^{-\frac{1}{2}} \|u^\varepsilon\|_Y \max \{ C_1^{-\frac{1}{2}}, C_0^{\frac{1}{2}} C_R \}.$$

Moreover, there exists $h_3 = h_3(\varepsilon) > 0$ such that for $h \leq \min\{h_0, h_2, h_3\}$

$$(4.33) \quad \|u^\varepsilon - u_h^\varepsilon\|_{L^2} \leq C_8 \left(C_2 h^{\ell+r-4} \|u^\varepsilon\|_{H^\ell} + L(h) C_7 h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right),$$

where

$$h_3 = C (C_7 \delta^{-1} \|u^\varepsilon\|_{H^\ell})^{\frac{1}{2-\ell}}, \quad C_8 = CC_7 C_R, \quad r = \min\{p, k+1\}.$$

PROOF. Let $\rho_1 := 2C_6h^{\ell-2}\|u^\varepsilon\|_{H^\ell}$, and note that for $h \leq h_2$, there holds $\rho_1 \leq \rho_0$. Thus for $h \leq \min\{h_0, h_2\}$ and noting $h_2 \leq h_1$, we use Lemmas 4.5 and 4.6 to conclude that for any $v_h \in \mathbb{B}_h(\rho_1)$,

$$\begin{aligned} \|\mathcal{I}^h u^\varepsilon - T_h(v_h)\|_{H^2} &\leq \|\mathcal{I}^h u^\varepsilon - T_h(\mathcal{I}^h u^\varepsilon)\|_{H^2} + \|T_h(\mathcal{I}^h u^\varepsilon) - T_h(v_h)\|_{H^2} \\ &\leq C_6 h^{\ell-2} \|u^\varepsilon\|_{H^\ell} + \frac{1}{2} \|\mathcal{I}^h u^\varepsilon - v_h\|_{H^2} \\ &\leq \frac{\rho_1}{2} + \frac{\rho_1}{2} = \rho_1. \end{aligned}$$

Hence, T_h maps $\mathbb{B}_h(\rho_1)$ into $\mathbb{B}_h(\rho_1)$. Since T_h is continuous and a contraction mapping in $\mathbb{B}_h(\rho_1)$, by Banach's Fixed Point Theorem [42] T_h has a unique fixed point $u_h^\varepsilon \in \mathbb{B}_h(\rho_1)$, which is the unique solution to (4.3). To derive the error estimate (4.32), we use the triangle inequality to obtain

$$\begin{aligned} \|u^\varepsilon - u_h^\varepsilon\|_{H^2} &\leq \|u^\varepsilon - \mathcal{I}^h u^\varepsilon\|_{H^2} + \|\mathcal{I}^h u^\varepsilon - u_h^\varepsilon\|_{H^2} \\ &\leq C h^{\ell-2} \|u^\varepsilon\|_{H^\ell} + \rho_1 \leq C_7 h^{\ell-2} \|u^\varepsilon\|_{H^\ell}. \end{aligned}$$

To obtain the L^2 error estimate (4.33), we start with the error equation:

$$\langle \Delta e^\varepsilon, \Delta v_h \rangle + \langle F(u^\varepsilon) - F(u_h^\varepsilon), v_h \rangle = 0 \quad \forall v_h \in V_0^h,$$

where $e^\varepsilon := u^\varepsilon - u_h^\varepsilon$. Using the mean value theorem, we obtain

$$(4.34) \quad \langle \Delta e^\varepsilon, \Delta v_h \rangle + \langle F'[y_h](e^\varepsilon), v_h \rangle = 0 \quad \forall v_h \in V_0^h,$$

where $y_h = u^\varepsilon - \gamma e^\varepsilon$ for some $\gamma \in [0, 1]$. Again, we have abused the notation of y_h , defining it differently in different proofs.

Next, let $w \in H^p(\Omega) \cap V_0$ ($p > 2$) be the solution to the following auxiliary problem:

$$\langle (G'_\varepsilon[u^\varepsilon])^*(w), z \rangle = (e^\varepsilon, z) \quad \forall z \in V_0,$$

with

$$(4.35) \quad \|w\|_{H^p} \leq C_R \|e^\varepsilon\|_{L^2}.$$

Using (4.34), we then have for any $w_h \in V_0^h$

$$\begin{aligned} (4.36) \quad \|e^\varepsilon\|_{L^2}^2 &= \langle (G'_\varepsilon[u^\varepsilon])^*(w), e^\varepsilon \rangle \\ &= \langle G'_\varepsilon[u^\varepsilon](e^\varepsilon), w \rangle \\ &= a^\varepsilon(e^\varepsilon, w) \\ &= a^\varepsilon(e^\varepsilon, w - w_h) + \varepsilon \langle \Delta e^\varepsilon, \Delta w_h \rangle + \langle F'[u^\varepsilon](e^\varepsilon), w_h \rangle \\ &= a^\varepsilon(e^\varepsilon, w - w_h) + \langle (F'[u^\varepsilon] - F'[y_h])(e^\varepsilon), w_h \rangle \\ &\leq CC_2 \|e^\varepsilon\|_{H^2} \|w - w_h\|_{H^2} + \|F'[u^\varepsilon] - F'[y_h]\|_{VV^*} \|e^\varepsilon\|_{H^2} \|w_h\|_{H^2}. \end{aligned}$$

Then by (4.32) for $h \leq h_3$

$$\|u^\varepsilon - y_h\|_{H^2} = \gamma \|e^\varepsilon\|_{H^2} \leq \delta.$$

Therefore, setting $w_h = \mathcal{I}_h w$ in (4.36), we have for $h \leq \min\{h_0, h_2, h_3\}$,

$$\begin{aligned} \|e^\varepsilon\|_{L^2}^2 &\leq C \left(C_2 h^{r-2} \|e^\varepsilon\|_{H^2} + L(h) \|e^\varepsilon\|_{H^2}^2 \right) \|w\|_{H^p} \\ &\leq CC_R \left(C_2 h^{r-2} \|e^\varepsilon\|_{H^2} + L(h) \|e^\varepsilon\|_{H^2}^2 \right) \|e^\varepsilon\|_{L^2}. \end{aligned}$$

Thus,

$$\begin{aligned}\|e^\varepsilon\|_{L^2} &\leq CC_R \left(C_2 h^{r-2} \|e^\varepsilon\|_{H^2} + L(h) \|e^\varepsilon\|_{H^2}^2 \right) \\ &\leq CC_7 C_R \left(C_2 h^{\ell+r-4} \|u^\varepsilon\|_{H^\ell} + L(h) C_7 h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right).\end{aligned}$$

□

Remark 4.8. (a) Noting $2\ell-4 > \ell$ for $\ell \geq 4$ and $k > 4$, Theorem 4.7 requires $p \geq 4$ to obtain optimal order error estimates in the L^2 -norm. This regularity condition is expected provided that the domain Ω is smooth and solution u^ε is sufficiently regular.

(b) If $(G'_\varepsilon(v))^*$ is coercive on V_0 , that is $C_0 = 0$ in the inequality (4.5), then $C_7 = CC_1^{-1} \|u^\varepsilon\|_Y$ in the error bound (4.32). Furthermore, it is expected that $C_1 = O(\varepsilon)$ in such cases, and therefore (4.32) reads

$$\|u^\varepsilon - u_h^\varepsilon\|_{H^2} \leq C \varepsilon^{-1} h^{\ell-2} \|u^\varepsilon\|_Y \|u^\varepsilon\|_{H^\ell}.$$

(c) We note that the constants C_2, C_7, C_8 appeared in the error bounds of Theorem 4.7 all depend on some negative powers of ε , which is expected. The dependence of C_2, C_7, C_8 on ε^{-1} we derived are the worst-case scenarios, they are far from being sharp (in particular, in the 3-D case) although the proved convergence rates in h are optimal. In Section 6 we shall present a detailed numerical study about the sharpness of the dependence of the error bounds on ε^{-1} . Our numerical experiments suggest that the error bounds only grow in ε^{-1} in some small power orders, which are considerably better than the theoretical estimates indicate.

Mixed finite element approximations

The goal of this chapter is to construct and analyze a family of Hermann-Miyoshi mixed finite element methods for general fully nonlinear second order problem (2.7)–(2.8) based on the vanishing moment method (2.9)–(2.11)₃. The mixed formulation is based on rewriting (2.9) as a system of two second order PDEs by introducing an additional variable. By decoupling (2.9) as a system, we are able to approximate (2.9)–(2.11)₃ using only C^0 finite elements, opposed to C^1 finite elements used in Chapter 4, which can be computational expensive and complicated.

We note that the theory of mixed finite element methods, such as Hermann-Miyoshi mixed methods, has been extensively developed in the seventies and eighties for biharmonic problems in two dimensions (cf. [22, 13]). It is straightforward to formulate these methods for the fourth order quasilinear PDE (2.9) in two and three dimensions. Although it is a simple task to define mixed finite element methods for problem (2.9)–(2.11)₃, proving existence of solutions and obtaining convergence rates are quite difficult. As is now well-known, proving existence and deriving error estimates for mixed methods relies heavily on the so-called inf-sup condition, and naturally, this is the starting point in our analysis. However, due to the strong nonlinearity in (2.9), the inf-sup condition is not sufficient for our purposes, and therefore, we must look for other techniques to obtain existence, uniqueness, and error estimates. To this end, we use a combined fixed-point and linearization technique that is in the same spirit as in the previous chapter.

The chapter is organized as follows. In Section 5.1, we define the mixed formulation of (2.9)–(2.11)₃, and then define the Hermann-Miyoshi mixed finite element method based upon this formulation. We then make certain structure assumptions on the nonlinear differential operator F , which will be used frequently in the analysis of the mixed finite element method. The assumptions are generally mild and are very similar to those in Chapter 4. In Section 5.2, we prove convergence results of the mixed finite element method for the linearized problem (4.7)–(4.9). In Section 5.3 we obtain our main results, where we obtain existence and uniqueness for the proposed Hermann-Miyoshi mixed finite element method and also derive error estimates.

5.1. Formulation of mixed finite element methods

There are several popular mixed formulations for fourth order problems. However, since the Hessian matrix appears in (2.9) in a nonlinear fashion, we cannot use Δu^ε as an additional variable. This observation then rules out the family of Ciarlet-Raviart mixed finite element methods. On the other hand, this observation motivates us to try Hermann-Miyoshi mixed elements which use $\sigma^\varepsilon := D^2 u^\varepsilon$ as

an additional unknown, and so, in this chapter, we will only focus on developing Hermann-Miyoshi type mixed methods for problem (2.9)–(2.11)₃.

In addition to the notation introduced in Section 1.2, we also define the following space notation:

$$\begin{aligned} Q &:= H^1(\Omega), & Q_0 &:= H_0^1(\Omega), \\ Q_g &:= \{v \in Q; v|_{\partial\Omega} = g\}, & W &:= \{\mu \in Q^{n \times n}; \mu_{ij} = \mu_{ji}\}, \\ W_0 &:= \{\mu \in W; \mu\nu \cdot \nu|_{\partial\Omega} = 0\}, & W_\varepsilon &:= \{\mu \in W; \mu\nu \cdot \nu|_{\partial\Omega} = \varepsilon\}. \end{aligned}$$

Recall that we use Greek letters to represent tensor functions and Roman letters to represent scalar functions throughout the paper.

To define the mixed variational formulation for problem (2.9)–(2.11)₃, we rewrite the PDE into a system of two second order equations as follows:

$$(5.1) \quad \sigma^\varepsilon - D^2 u^\varepsilon = 0,$$

$$(5.2) \quad \varepsilon \Delta \text{tr}(\sigma^\varepsilon) + F(\sigma^\varepsilon, u^\varepsilon) = 0,$$

where $F(\sigma^\varepsilon, u^\varepsilon)$ is defined in (1.13).

Testing (5.1) with $\mu \in W_0$, we get

$$(5.3) \quad (\sigma^\varepsilon, \mu) + (\text{div}(\mu), \nabla u^\varepsilon) = \sum_{i=1}^{n-1} \left\langle \mu\nu \cdot \tau_i, \frac{\partial g}{\partial \tau_i} \right\rangle_{\partial\Omega},$$

where $\{\tau_1(x), \dots, \tau_{n-1}(x)\}$ denotes the standard basis of the tangent space to $\partial\Omega$ at x , and

$$(\sigma^\varepsilon, \mu) = \int_{\Omega} \sigma^\varepsilon : \mu \, dx = \sum_{i,j=1}^n \int_{\Omega} \sigma_{ij}^\varepsilon \mu_{ij} \, dx.$$

Next, multiplying (5.2) with $w \in Q_0$ and integrating over Ω gives us

$$(5.4) \quad -\varepsilon (\text{div}(\sigma^\varepsilon), \nabla w) + (F(\sigma^\varepsilon, u^\varepsilon), w) = 0.$$

Based on (5.3)–(5.4), we define the mixed formulation of (2.9)–(2.11)₃ as follows: find $(\sigma^\varepsilon, u^\varepsilon) \in W_\varepsilon \times Q_g$ such that

$$(5.5) \quad (\sigma^\varepsilon, \kappa) + b(\kappa, u^\varepsilon) = G(\kappa) \quad \forall \kappa \in W_0,$$

$$(5.6) \quad b(\sigma^\varepsilon, v) - \varepsilon^{-1} c(\sigma^\varepsilon, u^\varepsilon, v) = 0 \quad \forall v \in Q_0,$$

where for $\mu \in W$, $v, w \in Q$

$$\begin{aligned} b(\mu, v) &:= (\text{div}(\mu), \nabla v), & c(\mu, w, v) &:= (F(\mu, w), v), \\ (5.7) \quad G(\mu) &:= \sum_{i=1}^{n-1} \left\langle \mu\nu \cdot \tau_i, \frac{\partial g}{\partial \tau_i} \right\rangle_{\partial\Omega}. \end{aligned}$$

Next, let \mathcal{T}_h be a quasiuniform triangular or quadrilateral partition of Ω if $n = 2$, and tetrahedral or hexahedra mesh if $n = 3$ parameterized by $h \in (0, 1)$. Let $Q^h \subset Q$ be the Lagrange finite element space consisting of globally continuous, piecewise polynomials of degree k (≥ 2) associated with the mesh \mathcal{T}_h .

We then define the following finite element spaces:

$$\begin{aligned} Q_0^h &:= Q^h \cap Q_0, & Q_g^h &:= Q^h \cap Q_g, \\ W_0^h &:= [Q^h]^{n \times n} \cap W_0, & W_\varepsilon^h &:= [Q^h]^{n \times n} \cap W_\varepsilon, \end{aligned}$$

and define the norms $\|(\cdot, \cdot)\|_\varepsilon$, $\|(\cdot, \cdot)\|_\varepsilon : W \times Q \mapsto \mathbf{R}^+$ such that for any $(\mu, v) \in W \times Q$,

$$\begin{aligned} \|(\mu, v)\|_\varepsilon &:= \|\mu\|_{L^2} + K_1^{\frac{1}{2}} \varepsilon^{-\frac{1}{2}} \|v\|_{H^1}, \\ \|(\mu, v)\|_\varepsilon &:= h \|\mu\|_{H^1} + \|(\mu, v)\|_\varepsilon, \end{aligned}$$

and K_1 is defined by [B2] below.

Based on (5.5)–(5.6), we define the Herman-Miyoshi-type mixed finite element method as follows: find $(\sigma_h^\varepsilon, u_h^\varepsilon) \in W_\varepsilon^h \times Q_g^h$ such that

$$(5.8) \quad (\sigma_h^\varepsilon, \kappa_h) + b(\kappa_h, u_h^\varepsilon) = G(\kappa_h) \quad \forall \kappa_h \in W_0^h,$$

$$(5.9) \quad b(\sigma_h^\varepsilon, z_h) - c(\sigma_h^\varepsilon, u_h^\varepsilon, z_h) = 0 \quad \forall z_h \in Q_0^h.$$

The main goal of this chapter is to prove existence and uniqueness for problem (5.8)–(5.9) and to also derive error estimates for $\sigma^\varepsilon - \sigma_h^\varepsilon$ and $u^\varepsilon - u_h^\varepsilon$. As a first step, we state the following inf-sup condition for the finite element pair (W_0^h, Q_0^h) . The proof can be found in [38, 61].

Lemma 5.1. *For every $w_h \in Q_0^h$, there exists $C > 0$ independent of h , such that*

$$(5.10) \quad \sup_{\mu_h \in W_0^h} \frac{b(\mu_h, w_h)}{\|\mu_h\|_{H^1}} \geq C \|w_h\|_{H^1}.$$

Remark 5.2. By [35, Proposition 1], Lemma 5.1 implies that there exists a linear operator $\Pi^h : W \mapsto W^h$ such that

$$(5.11) \quad b(\mu - \Pi^h \mu, w_h) = 0 \quad \forall w_h \in Q_0^h,$$

and for $\mu \in W \cap [H^s(\Omega)]^{n \times n}$, $s \geq 1$, there holds

$$(5.12) \quad \|\mu - \Pi^h \mu\|_{H^j} \leq Ch^{\ell-j} |\mu|_{H^\ell} \quad j = 0, 1, \quad 1 \leq \ell \leq \min\{s, k+1\}.$$

Next, we assume the following structure conditions on the nonlinear differential operator F , which play an important role in our analysis.

Assumption (B)

[B1] There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$, there exists a locally unique solution to (2.9)–(2.11)₃ with $u^\varepsilon \in H^s(\Omega)$ ($s \geq 3$).

[B2] The operator $(G'_\varepsilon[\sigma^\varepsilon, u^\varepsilon])^*$ (the adjoint of $G'_\varepsilon[\sigma^\varepsilon, u^\varepsilon]$) is an isomorphism from $H^2(\Omega) \cap H_0^1(\Omega)$ to $(H^2(\Omega) \cap H_0^1(\Omega))^*$. That is for any $\varphi \in (H^2(\Omega) \cap H_0^1(\Omega))^*$, there exists $v \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$(5.13) \quad \left\langle (G'_\varepsilon[\sigma^\varepsilon, u^\varepsilon])^*(D^2 v, v), w \right\rangle = \langle \varphi, w \rangle \quad \forall w \in H^2(\Omega) \cap H_0^1(\Omega).$$

Furthermore, there exists positive constants $K_0 = K_0(\varepsilon)$, $K_1 = K_1(\varepsilon)$, such that the following Gårding inequality holds¹:

$$(5.14) \quad \langle F'[\sigma^\varepsilon, u^\varepsilon](D^2 v, v), v \rangle \geq K_1 \|v\|_{H^1}^2 - K_0 \|v\|_{L^2}^2 \quad \forall v \in Q_0,$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing of Q_0 and Q_0^* . Also, there exists $K_2 = K_2(\varepsilon) > 0$ such that

$$\|F'[\sigma^\varepsilon, u^\varepsilon]\|_{QQ^*} \leq K_2,$$

¹See Remark 5.3(d) for an interpretation.

where

$$\begin{aligned} \|F'[\sigma^\varepsilon, u^\varepsilon]\|_{QQ^*} &:= \sup_{v \in Q_0} \frac{\|F'[\sigma^\varepsilon, u^\varepsilon](D^2v, v)\|_{H^{-1}}}{\|v\|_{H^1}} \\ &:= \sup_{v \in Q_0} \sup_{w \in Q_0} \frac{\langle F'[\sigma^\varepsilon, u^\varepsilon](D^2v, v), w \rangle}{\|v\|_{H^1} \|w\|_{H^1}}. \end{aligned}$$

Moreover, there exists $p \geq 3$ and $K_{R_m} > 0$, ($m = 0, 1$) such that if $\varphi \in H^{-m}(\Omega)$ and $v \in V_0$ satisfies (5.13), then $v \in H^{p-m}(\Omega)$ and

$$\|v\|_{H^{p-m}} \leq K_{R_m} \|\varphi\|_{H^{-m}}.$$

[B3] There exists Banach spaces X, Y with a functional

$$\|(\cdot, \cdot)\|_{X \times Y} : X \times Y \mapsto \mathbf{R}^+,$$

and a constant $C > 0$ such that for all $\omega \in X$, $y \in Y$, $\chi \in W$, $v \in Q$

$$\|F'[\omega, y](\chi, v)\|_{H^{-1}} \leq C \|(\omega, y)\|_{X \times Y} (\|\chi\|_{L^2} + \|v\|_{H^1}).$$

Furthermore, $\|(\cdot, \cdot)\|_{X \times Y}$ is well-defined and finite on $W^h \times Q^h$.

[B4] There exists a constant $K_3 = K_3(\varepsilon) > 0$ such that

$$\left\| (\Pi^h \sigma^\varepsilon - \gamma \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon - \gamma u^\varepsilon) \right\|_{X \times Y} \leq K_3(\varepsilon) \quad \forall \gamma \in [0, 1].$$

where $\mathcal{I}^h u^\varepsilon \in Q_g^h$ is the finite element interpolant of u^ε .

[B5] There exists a constant $\delta = \delta(\varepsilon) \in (0, 1)$, such that for any $(\mu_h, v_h) \in W_\varepsilon^h \times Q_g^h$ with $\|(\Pi^h \sigma^\varepsilon - \mu_h, \mathcal{I}^h u^\varepsilon - v_h)\|_\varepsilon \leq \delta$, there holds $\forall (\kappa_h, z_h) \in W^h \times Q^h$

$$\begin{aligned} &\sup_{w_h \in Q^h} \frac{\langle (F'[\sigma^\varepsilon, u^\varepsilon] - F'[\mu_h, v_h])(\kappa_h, z_h), w_h \rangle}{\|w_h\|_{H^1}} \\ &\leq R(h) (\|\sigma^\varepsilon - \mu_h\|_{L^2} + \|u^\varepsilon - v_h\|_{H^1}) \|(\kappa_h, z_h)\|_\varepsilon, \end{aligned}$$

where $R(h) = R(\varepsilon, h)$ may depend on ε and h and $R(h) = o(h^{2-\ell})$.

[B6] There exists $K_G = K_G(\varepsilon) > 0$ and $\alpha > 0$ such that for any

$$(\chi_h, v_h) \in \mathbb{T}_h := \{(\kappa_h, z_h) \in W_0^h \times Q_0; (\kappa_h, \chi_h) + b(\chi_h, z_h) = 0 \ \forall \chi_h \in W_0^h\},$$

there holds

$$\|F'[\sigma^\varepsilon, u^\varepsilon](\chi_h - D^2v_h, 0)\|_{H^{-1}} \leq K_G h^\alpha \|(\chi_h, v_h)\|_\varepsilon.$$

Remark 5.3. (a) We made an effort in our presentation to state assumptions in this section that resemble those in the previous chapter, where conforming finite element methods for (2.9)–(2.11)₁ were studied. It is clear that conditions [B1]–[B6] are similar, but slightly stronger than conditions [A1]–[A5]. For example, the inequality (5.14) suggests that the operator $-F'[u^\varepsilon]$ is *uniformly* elliptic, which rules out degenerate problems. However, assumptions [B1]–[B6] are still not very restrictive, and we will show in Chapter 6 that many well-known fully nonlinear second order differential operators satisfy these requirements. We also show a simple trick at the end of the chapter which makes it possible to incorporate degenerate elliptic PDEs (i.e. $K_1=0$) into the theory.

(b) We note that by definition of σ^ε , G'_ε , and F' (see Section 1.2)

$$\begin{aligned} G'_\varepsilon[\sigma^\varepsilon, u^\varepsilon](D^2v, v) &= G'_\varepsilon[D^2u^\varepsilon, u^\varepsilon](D^2v, v) = G'_\varepsilon[u^\varepsilon](v), \\ F'[\sigma^\varepsilon, u^\varepsilon](D^2v, v) &= F'[D^2u^\varepsilon, u^\varepsilon](D^2v, v) = F'[u^\varepsilon](v). \end{aligned}$$

It then seems redundant to write $G'_\varepsilon[\sigma^\varepsilon, u^\varepsilon](D^2v, v)$ and $F'[\sigma^\varepsilon, u^\varepsilon](D^2v, v)$ instead of $G'_\varepsilon[u^\varepsilon](v)$ and $F'[u^\varepsilon](v)$. However, this (longer) short-hand notation naturally fits into the mixed method framework, and makes the subsequent analysis easier to follow.

(c) It is obvious that assumption [B1] is needed, and this assumption is actually the same as [A1]; we include it again for consistency and standardization.

(d) Assumption [B2] is a natural extension of [A2], and F is expected to satisfy these conditions provided that $-F$ is uniformly elliptic at u^ε , and $\partial\Omega$ is sufficiently regular. We note that (5.14) needs to be understood with care because of the special notation we use. The left-hand side should be understood in the distributional sense. To derive the inequality, an integration by parts must be used on the second order derivative term. Also, Remark 4.8 gives heuristic estimates for the constants K_{R_0} and K_{R_1} in terms of ε .

(e) By the standard interpolation theory and (5.12), we have

$$(5.15) \quad h^{-1} \|\sigma^\varepsilon - \Pi^h \sigma^\varepsilon\|_{L^2} + \|\sigma^\varepsilon - \Pi^h \sigma^\varepsilon\|_{H^1} \leq Ch^{\ell-1} \|\sigma^\varepsilon\|_{H^\ell}.$$

$$(5.16) \quad h^{-1} \|u^\varepsilon - \mathcal{I}^h u^\varepsilon\|_{L^2} + \|u^\varepsilon - \mathcal{I}^h u^\varepsilon\|_{H^1} \leq Ch^{\ell-1} \|u^\varepsilon\|_{H^\ell}.$$

(f) Condition [B5], which is used in the fixed-point argument, states that F' is in some sense locally Lipschitz near $(\sigma^\varepsilon, u^\varepsilon)$.

(g) Clearly, if $(\kappa, z) \in W_0 \times Q_0$ satisfy

$$(\kappa, \chi) + b(\chi, z) = 0 \quad \forall \chi \in W_0,$$

then $D^2z = \kappa$ in a weak sense. However, if $(\kappa_h, z_h) \in \mathbb{T}_h$, the analogous equality $D^2z_h = \kappa_h$ is not necessarily true. Assumption [B6] indicates that the discrepancy between κ_h and D^2z_h under the image of $F'[\sigma^\varepsilon, u^\varepsilon]$ is small. However, in what follows, we show that assumption [B6] holds with $\alpha = 1$ if F is sufficiently smooth at the solution $(\sigma^\varepsilon, u^\varepsilon)$.

Proposition 5.4. *Suppose*

$$\frac{\partial F(\sigma^\varepsilon, u^\varepsilon)}{\partial r_{ij}} \in L^\infty(\Omega) \cap W^{1, \frac{6}{5}}(\Omega) \quad i, j = 1, 2, \dots, n.$$

Then assumption [B6] holds with $\alpha = 1$ and

$$K_G = C \left(\max_{1 \leq i, j \leq n} \left\| \frac{\partial F(\sigma^\varepsilon, u^\varepsilon)}{\partial r_{ij}} \right\|_{L^\infty} + \max_{1 \leq i, j \leq n} \left\| \frac{\partial F(\sigma^\varepsilon, u^\varepsilon)}{\partial r_{ij}} \right\|_{W^{1, \frac{6}{5}}} \right).$$

PROOF. For any $z \in Q_0$, define λ^ε such that

$$\lambda_{ij}^\varepsilon = \frac{\partial F(\sigma^\varepsilon, u^\varepsilon)}{\partial r_{ij}} z.$$

Then using the property (5.11), we have for any $(\chi_h, v_h) \in \mathbb{T}_h$

$$\begin{aligned} \langle F'[\sigma^\varepsilon, u^\varepsilon](\chi_h - D^2v_h, 0), z \rangle &= \langle \chi_h - D^2v_h, \lambda^\varepsilon \rangle = (\chi_h, \lambda^\varepsilon) + b(\lambda^\varepsilon, v_h) \\ &= (\chi_h, \lambda^\varepsilon) + b(\Pi^h \lambda^\varepsilon, v_h) = (\chi_h, \lambda^\varepsilon - \Pi^h \lambda^\varepsilon) \\ &\leq \|(\chi_h, v_h)\|_\varepsilon \|\lambda^\varepsilon - \Pi^h \lambda^\varepsilon\|_{L^2}. \end{aligned}$$

Next, by (5.12), the definition of λ^ε , and the product rule, we have

$$\begin{aligned} & \|\lambda^\varepsilon - \Pi^h \lambda^\varepsilon\|_{L^2} \\ & \leq Ch \left(\|\nabla z\|_{L^2} \max_{1 \leq i, j \leq n} \left\| \frac{\partial F(\sigma^\varepsilon, u^\varepsilon)}{\partial r_{ij}} \right\|_{L^\infty} + \|z\|_{L^6} \max_{1 \leq i, j \leq n} \left\| \frac{\partial F(\sigma^\varepsilon, u^\varepsilon)}{\partial r_{ij}} \right\|_{W^{1, \frac{6}{5}}} \right). \end{aligned}$$

Therefore, by Poincaré's inequality and a Sobolev inequality

$$\begin{aligned} & \|\lambda^\varepsilon - \Pi^h \lambda^\varepsilon\|_{L^2} \\ & \leq Ch \left(\max_{1 \leq i, j \leq n} \left\| \frac{\partial F(\sigma^\varepsilon, u^\varepsilon)}{\partial r_{ij}} \right\|_{L^\infty} + \max_{1 \leq i, j \leq n} \left\| \frac{\partial F(\sigma^\varepsilon, u^\varepsilon)}{\partial r_{ij}} \right\|_{W^{1, \frac{6}{5}}} \right) \|z\|_{H^1}. \end{aligned}$$

The result follows from the above inequality. \square

5.2. Linearization and its mixed finite element approximations

To derive existence, uniqueness, and the desired error estimates for the mixed finite element method (5.8)–(5.9), we must first study the mixed finite element approximations of (4.7)–(4.9), but with an alternative boundary condition:

$$(5.17) \quad G'_\varepsilon[u^\varepsilon](v) = \varphi \quad \text{in } \Omega,$$

$$(5.18) \quad v = 0 \quad \text{on } \partial\Omega,$$

$$(5.19) \quad D^2 v \nu \cdot \nu = 0 \quad \text{on } \partial\Omega,$$

where $\varphi \in Q_0^*$ is some given function. Using arguments similar to the proof of Theorem 4.2, we conclude that there exists a unique solution $v \in H^2(\Omega) \cap H_0^1(\Omega)$ to (5.17)–(5.19).

To introduce a mixed formulation for (5.17)–(5.19), we rewrite the fourth order PDE (5.17) as the following system of two second order PDEs:

$$(5.20) \quad \chi - D^2 v = 0,$$

$$(5.21) \quad \varepsilon \Delta \text{tr}(\chi) + F'[\sigma^\varepsilon, u^\varepsilon](D^2 v, v) = \varphi,$$

where $\text{tr}(\chi)$ denotes the trace of χ .

The mixed variational formulation of (5.17)–(5.19) is then defined as follows: find $(\chi, v) \in W_0 \times Q_0$ such that

$$(5.22) \quad (\chi, \mu) + b(\mu, v) = 0 \quad \forall \mu \in W_0,$$

$$(5.23) \quad b(\chi, w) - \varepsilon^{-1} d(u^\varepsilon; v, w) = -\varepsilon^{-1} \langle \varphi, w \rangle \quad \forall w \in Q_0,$$

where for $v, w \in Q$

$$d(u^\varepsilon; v, w) := \langle F'[\sigma^\varepsilon, u^\varepsilon](D^2 v, v), w \rangle$$

Remark 5.5. We note again that the right-hand side of $d(\cdot; \cdot, \cdot)$ should be understood in the distributional sense.

5.2.1. Mixed finite element approximation of linearized problem. Based on the variational formulation (5.22)–(5.23), we define the mixed finite element method for (5.17)–(5.19) as seeking $(\chi_h, w_h) \in W_0^h \times Q_0^h$ such that

$$(5.24) \quad (\chi_h, \mu_h) + b(\mu_h, v_h) = 0 \quad \forall \mu_h \in W_0^h,$$

$$(5.25) \quad b(\chi_h, w_h) - \varepsilon^{-1} d(u^\varepsilon; v_h, w_h) = -\varepsilon^{-1} \langle \varphi, w_h \rangle \quad \forall w_h \in Q_0^h.$$

Our objective in this section is to prove existence and uniqueness for problem (5.24)–(5.25) and then to derive error estimates in various norms.

Theorem 5.6. *Suppose assumptions [B1]–[B2] hold. Let $v \in H^s(\Omega)$ ($s \geq 3$) be the unique solution to (5.17)–(5.19) and $\chi = D^2 v$. Then there exists $h_0 = h_0(\varepsilon) > 0$ such that for $h \leq h_0$, there exists a unique solution $(\chi_h, w_h) \in W_0^h \times Q_0^h$ to problem (5.24)–(5.25), where*

$$h_0 = \begin{cases} C \left(\min \left\{ (K_0 K_1^{-1} K_2^2 K_{R_0}^2)^{\frac{1}{2-2r}}, (K_0 K_{R_0}^2 \varepsilon)^{\frac{1}{4-2r}} \right\} \right) & \text{if } K_0 \neq 0, \\ 1 & \text{if } K_0 = 0, \end{cases}$$

$$r = \min\{p, k+1\}.$$

Furthermore, there hold the following error estimates:

$$(5.26) \quad |||(\chi - \chi_h, v - v_h)|||_\varepsilon \leq Ch^{\ell-2} (K_4 h + 1) \|v\|_{H^\ell},$$

$$(5.27) \quad \|v - v_h\|_{L^2} \leq K_5 h^{\ell+r-4} (K_4 h + 1) \|v\|_{H^\ell}.$$

where

$$K_4 = C \max\{K_1^{-\frac{1}{2}} K_2 \varepsilon^{-\frac{1}{2}}, K_0^{\frac{1}{2}} K_{R_0} \varepsilon^{\frac{1}{2}}\}, \quad K_5 = C K_1^{-\frac{1}{2}} K_2 K_{R_0} \varepsilon^{\frac{1}{2}},$$

$$\ell = \min\{s, k+1\}.$$

PROOF. We first start by showing that the error estimates (5.26)–(5.27) hold in the case that there does exist a solution to (5.24)–(5.25).

Let $\mathcal{I}^h v$ denote the standard finite element interpolant of v in Q_0^h . Then using (5.11), we have for all $(\mu_h, w_h) \in W_0^h \times Q_0^h$,

$$(5.28) \quad (\chi_h - \Pi^h \chi, \mu_h) + b(\mu_h, v_h - \mathcal{I}^h v) = (\chi - \Pi^h \chi, \mu_h) + b(\mu_h, v - \mathcal{I}^h v),$$

$$(5.29) \quad b(\chi_h - \Pi^h \chi, w_h) - \varepsilon^{-1} d(u^\varepsilon; v_h - \mathcal{I}^h v, w_h) \\ = \varepsilon^{-1} d(u^\varepsilon; \mathcal{I}^h v - v, w_h).$$

Setting $\mu_h = \chi_h - \Pi^h \chi$ and $w_h = v_h - \mathcal{I}^h v$ and subtracting (5.29) from (5.28) yields

$$(5.30) \quad (\chi_h - \Pi^h \chi, \chi_h - \Pi^h \chi) + \varepsilon^{-1} d(u^\varepsilon; v_h - \mathcal{I}^h v, v_h - \mathcal{I}^h v) \\ = (\chi - \Pi^h \chi, \chi_h - \Pi^h \chi) + b(\chi_h - \Pi^h \chi, v - \mathcal{I}^h v) \\ + \varepsilon^{-1} d(u^\varepsilon; v - \mathcal{I}^h v, v_h - \mathcal{I}^h v).$$

Thus, by assumption [B2],

$$\begin{aligned} & \|(\chi - \Pi^h \chi, v_h - \mathcal{I}^h v)\|_\varepsilon^2 \\ & \leq \|\chi - \Pi^h \chi\|_{L^2} \|\chi_h - \Pi^h \chi\|_{L^2} + \|\operatorname{div}(\chi_h - \Pi^h \chi)\|_{L^2} \|\nabla(v - \mathcal{I}^h v)\|_{L^2} \\ & \quad + \varepsilon^{-1} \|F'[\sigma^\varepsilon, u^\varepsilon]\|_{QQ^*} \|v - \mathcal{I}^h v\|_{H^1} \|w_h\|_{H^1} + K_0 \varepsilon^{-1} \|v_h - \mathcal{I}^h v\|_{L^2}^2 \\ & \leq \|\chi - \Pi^h \chi\|_{L^2} \|\chi_h - \Pi^h \chi\|_{L^2} + Ch^{-1} \|\chi_h - \Pi^h \chi\|_{L^2} \|\nabla(v - \mathcal{I}^h v)\|_{L^2} \\ & \quad + K_2 \varepsilon^{-1} \|v - \mathcal{I}^h v\|_{H^1} \|v_h - \mathcal{I}^h v\|_{H^1} + K_0 \varepsilon^{-1} \|v_h - \mathcal{I}^h v\|_{L^2}^2, \end{aligned}$$

where we have used the inverse inequality in the last expression.

Using the Schwarz inequality, standard interpolation estimates, and rearranging terms, we have

$$\begin{aligned}
& \|(\chi_h - \Pi^h \chi, v_h - \mathcal{I}^h v)\|_\varepsilon^2 \\
& \leq C \left(\|\chi - \Pi^h \chi\|_{L^2}^2 + h^{-2} \|\nabla(v - \mathcal{I}^h v)\|_{L^2}^2 \right. \\
& \quad \left. + K_1^{-1} K_2^2 \varepsilon^{-1} \|v - \mathcal{I}^h v\|_{H^1}^2 + K_0 \varepsilon^{-1} \|v_h - \mathcal{I}^h v\|_{L^2}^2 \right) \\
& \leq C \left(h^{2\ell-4} (K_1^{-1} K_2^2 \varepsilon^{-1} h^2 + 1) \|v\|_{H^\ell}^2 + K_0 \varepsilon^{-1} \|v_h - \mathcal{I}^h v\|_{L^2}^2 \right),
\end{aligned}$$

which by an application of the triangle and inverse inequalities yields

$$\begin{aligned}
(5.31) \quad & \|(\chi - \chi_h, v - v_h)\|_\varepsilon^2 \\
& \leq C \left(h^{2\ell-4} (K_3^2 h^2 + 1) \|v\|_{H^\ell}^2 + K_0 \varepsilon^{-1} \|v - v_h\|_{L^2}^2 \right).
\end{aligned}$$

Continuing, we let $w \in Q_0 \cap H^p(\Omega)$ ($p \geq 3$) be the solution to the following auxiliary problem:

$$\begin{aligned}
(G'_\varepsilon[u^\varepsilon])^*(w) &= v - v_h & \text{in } \Omega, \\
D^2 w \nu \cdot \nu &= 0 & \text{on } \partial\Omega.
\end{aligned}$$

By assumption [B2], there exists such a solution and

$$(5.32) \quad \|w\|_{H^p} \leq K_{R_0} \|v - v_h\|_{L^2}.$$

Setting $\kappa = D^2 w \in [H^{p-2}(\Omega)]^{n \times n}$, it is easy to verify that (κ, w) satisfy

$$\begin{aligned}
(\kappa, \mu) + b(\mu, z) &= 0 & \forall \mu \in W_0, \\
b(\kappa, z) - \varepsilon^{-1} d^*(u^\varepsilon; w, z) &= \varepsilon^{-1} (v - v_h, z) & \forall z \in Q_0,
\end{aligned}$$

where $d^*(u^\varepsilon; \cdot, \cdot)$ denotes the adjoint of $d(u^\varepsilon; \cdot, \cdot)$, that is,

$$d^*(u^\varepsilon; v, w) = d(u^\varepsilon; w, v) \quad \forall v, w \in Q_0.$$

We also note that there hold the following Galerkin orthogonality:

$$\begin{aligned}
(\chi - \chi_h, \mu_h) + b(\mu_h, v - v_h) &= 0 & \forall \mu_h \in W_0^h, \\
b(\chi - \chi_h, w_h) - \varepsilon^{-1} d(u^\varepsilon; v - v_h, w_h) &= 0 & \forall w_h \in Q_0^h.
\end{aligned}$$

Thus, choosing $z = v - v_h$ we get

$$\begin{aligned}
\varepsilon^{-1} \|v - v_h\|_{L^2}^2 &= -b(\kappa, v - v_h) + \varepsilon^{-1} d^*(u^\varepsilon; w, v - v_h) \\
&= -b(\kappa - \Pi^h \kappa, v - v_h) + \varepsilon^{-1} d(u^\varepsilon; v - v_h, w) \\
&\quad - b(\Pi^h \kappa, v - v_h) \\
&= -b(\kappa - \Pi^h \kappa, v - \mathcal{I}^h v) + \varepsilon^{-1} d(u^\varepsilon; v - v_h, w) \\
&\quad + (\chi - \chi_h, \Pi^h \kappa) \\
&= -b(\kappa - \Pi^h \kappa, v - \mathcal{I}^h v) + \varepsilon^{-1} d(u^\varepsilon; v - v_h, w) \\
&\quad + (\chi - \chi_h, \kappa) + (\chi - \chi_h, \Pi^h \kappa - \kappa) \\
&= -b(\kappa - \Pi^h \kappa, v - \mathcal{I}^h v) + \varepsilon^{-1} d(u^\varepsilon; v - v_h, w) \\
&\quad - b(\chi - \chi_h, w) + (\chi - \chi_h, \Pi^h \kappa - \kappa) \\
&= -b(\kappa - \Pi^h \kappa, v - \mathcal{I}^h v) + \varepsilon^{-1} d(u^\varepsilon; v - v_h, w - \mathcal{I}^h w) \\
&\quad - b(\chi - \chi_h, w - \mathcal{I}^h w) + (\chi - \chi_h, \Pi^h \kappa - \kappa).
\end{aligned}$$

Therefore, using (5.32),

$$\begin{aligned}
\varepsilon^{-1} \|v - v_h\|_{L^2}^2 &\leq \|\operatorname{div}(\kappa - \Pi^h \kappa)\|_{L^2} \|\nabla(v - \mathcal{I}^h v)\|_{L^2} + K_2 \varepsilon^{-1} \|v - v_h\|_{H^1} \|w - \mathcal{I}^h w\|_{H^1} \\
&\quad + \|\operatorname{div}(\chi - \chi_h)\|_{L^2} \|\nabla(w - \mathcal{I}^h w)\|_{L^2} + \|\chi - \chi_h\|_{L^2} \|\Pi^h \kappa - \kappa\|_{L^2} \\
&\leq C \left(h^{\ell+r-4} \|\kappa\|_{H^{r-2}} \|v\|_{H^\ell} + K_2 \varepsilon^{-1} h^{r-1} \|w\|_{H^r} \|\nabla(v - v_h)\|_{L^2} \right. \\
&\quad \left. + h^{r-1} \|\operatorname{div}(\chi - \chi_h)\|_{L^2} \|w\|_{H^r} + h^{r-2} \|\chi - \chi_h\|_{L^2} \|\kappa\|_{H^{r-2}} \right) \\
&\leq CK_{R_0} \left(h^{\ell+r-4} \|v\|_{H^\ell} + K_2 \varepsilon^{-1} h^{r-1} \|\nabla(v - v_h)\|_{L^2} \right. \\
&\quad \left. + h^{r-1} \|\chi - \chi_h\|_{H^1} + h^{r-2} \|\chi - \chi_h\|_{L^2} \right) \|v - v_h\|_{L^2},
\end{aligned}$$

and hence

$$\begin{aligned}
(5.33) \quad \|v - v_h\|_{L^2}^2 &\leq CK_{R_0}^2 \varepsilon^2 \left(h^{2\ell+2r-8} \|v\|_{H^\ell}^2 + K_2^2 \varepsilon^{-2} h^{2r-2} \|\nabla(v - v_h)\|_{L^2}^2 \right. \\
&\quad \left. + h^{2r-2} \|\chi - \chi_h\|_{H^1}^2 + h^{2r-4} \|\chi - \chi_h\|_{L^2}^2 \right).
\end{aligned}$$

Using estimate (5.33) in (5.31) yields

$$\begin{aligned}
&\|(\chi - \chi_h, v - v_h)\|_\varepsilon^2 \\
&\leq C \left(h^{2\ell-4} (K_3^2 h^2 + 1) \|v\|_{H^\ell}^2 + \varepsilon^{-1} K_0 \|v - v_h\|_{L^2}^2 \right) \\
&\leq C \left(h^{2\ell-4} (K_4^2 h^2 + 1) \|v\|_{H^\ell}^2 \right. \\
&\quad \left. + K_0 K_{R_0}^2 \varepsilon \left[h^{2\ell+2r-8} \|v\|_{H^\ell}^2 + K_2^2 \varepsilon^{-2} h^{2r-2} \|\nabla(v - v_h)\|_{L^2}^2 \right. \right. \\
&\quad \left. \left. + h^{2r-2} \|\chi - \chi_h\|_{H^1}^2 + h^{2r-4} \|\chi - \chi_h\|_{L^2}^2 \right] \right).
\end{aligned}$$

It then follows that for $h \leq h_0$,

$$\begin{aligned} & \|\chi - \chi_h, v - v_h\|_\varepsilon^2 \\ & \leq C \left(h^{2\ell-4} (K_4^2 h^2 + 1) \|v\|_{H^\ell}^2 + K_0 K_{R_1}^2 \varepsilon h^{2\ell+2r-8} \|v\|_{H^\ell}^2 \right), \end{aligned}$$

and therefore

$$\begin{aligned} & \|\chi - \chi_h, v - v_h\|_\varepsilon \\ & \leq C \left\{ h^{\ell-2} (K_4 h + 1) \|v\|_{H^\ell} + K_0^{\frac{1}{2}} K_{R_0} \varepsilon^{\frac{1}{2}} h^{\ell+r-4} \|v\|_{H^\ell} \right\} \\ & \leq C h^{\ell-2} (K_4 h + 1) \|v\|_{H^\ell}, \end{aligned}$$

where we have used the fact that $r \geq 3$. Finally, (5.27) is obtained from (5.26) and (5.33).

So far, we have been working under the assumption that there exists a solution (χ_h, v_h) . However, using the Schatz's argument similar to the end of Theorem 4.3, we can conclude from (5.26)–(5.27) that (5.24)–(5.25) has a unique solution for $h \leq h_0$. \square

Remark 5.7. (a) To obtain optimal order error estimates in the L^2 -norm (5.27), we require $p \geq 4$ and $k \geq 3$ in the proof of Theorem 5.6.

(b) It is natural to ask why we use (5.21) instead of the alternative formulation

$$(5.34) \quad \varepsilon \Delta \text{tr}(\chi) + F'[\sigma^\varepsilon, u^\varepsilon](\chi, v) = \varphi.$$

As it turns out, it is advantageous to use (5.21) opposed to (5.34), as we now explain.

If we based the mixed finite element method on (5.34), the method would be similar to (5.24)–(5.25), but with $d(u^\varepsilon; v_h, w_h)$ replaced by $\hat{d}(u^\varepsilon; \chi_h, v_h, w_h)$, where

$$\hat{d}(u^\varepsilon; \chi_h, v_h, w_h) := \langle F'[\sigma^\varepsilon, u^\varepsilon](\chi_h, v_h), w_h \rangle.$$

Notice that by assumption [B2] (cf. (5.14)) there holds

$$d(u^\varepsilon; v_h, v_h) \geq K_1 \|v_h\|_{H^1}^2 - K_0 \|v_h\|_{L^2}^2 \quad \forall v_h \in Q_0^h.$$

However, an inequality such as this one does not hold for $\hat{d}(u^\varepsilon; \chi_h, v_h, v_h)$ even if $(\chi_h, v_h) \in \mathbb{T}_h$, where \mathbb{T}_h is defined in [B6].

However, if $(\chi_h, v_h) \in \mathbb{T}_h$, and if we define $\lambda^\varepsilon \in W_0$ such that

$$\lambda_{ij}^\varepsilon = \frac{\partial F}{\partial r_{ij}}(\sigma^\varepsilon, u^\varepsilon) v_h \quad i, j = 1, 2, \dots, n,$$

then

$$\begin{aligned} \hat{d}(u^\varepsilon; \chi_h, v_h, v_h) &= (\chi_h, \lambda^\varepsilon) + \langle F_p[\sigma^\varepsilon, u^\varepsilon](\chi_h, v_h), v_h \rangle + \langle F_z[\sigma^\varepsilon, u^\varepsilon](\chi_h, v_h), v_h \rangle \\ &= (\chi_h, \lambda^\varepsilon - \Pi^h \lambda^\varepsilon) - b(\lambda^\varepsilon, v_h) \\ &\quad + \langle F_p[\sigma^\varepsilon, u^\varepsilon](\chi_h, v_h), v_h \rangle + \langle F_z[\sigma^\varepsilon, u^\varepsilon](\chi_h, v_h), v_h \rangle, \end{aligned}$$

and after integrating by parts

$$\hat{d}(u^\varepsilon; \chi_h, v_h, v_h) = (\chi_h, \lambda^\varepsilon - \Pi^h \lambda^\varepsilon) + d(u^\varepsilon; v_h, v_h).$$

Thus, to obtain any coercivity from the alternative bilinear form $\hat{d}(u^\varepsilon; \cdot, \cdot, \cdot)$, we need to obtain bounds for $\|\lambda^\varepsilon - \Pi^h \lambda^\varepsilon\|_{L^2}$, adding more complexity to the proof

of Theorem 5.6. We also note the similarities of this derivation and the proof of Proposition 5.4.

5.3. Convergence analysis of mixed finite element methods

In this section, we give the main results of the chapter by proving there exists a unique solution to (5.8)–(5.9) and deriving error estimates of the numerical solution. First, we define the bilinear operator $\mathbf{M}_h : W_\varepsilon^h \times Q_g^h \mapsto W_\varepsilon^h \times Q_g^h$ such that for given $(\mu_h, w_h) \in W_\varepsilon^h \times Q_g^h$, $\mathbf{M}_h(\mu_h, w_h) := (M_h^{(1)}(\mu_h, w_h), M_h^{(2)}(\mu_h, w_h)) \in W_\varepsilon^h \times Q_g^h$ is the solution to the following problem:

$$(5.35) \quad (\mu_h - M_h^{(1)}(\mu_h, w_h), \kappa_h) + b(\kappa_h, w_h - M_h^{(2)}(\mu_h, w_h)) \\ = (\mu_h, \kappa_h) + b(\kappa_h, w_h) - G(\kappa_h) \quad \forall \kappa_h \in W_0^h,$$

$$(5.36) \quad b(\mu_h - M_h^{(1)}(\mu_h, w_h), z_h) - \varepsilon^{-1}d(u^\varepsilon; w_h - M_h^{(2)}(\mu_h, w_h), z_h) \\ = b(\mu_h, z_h) - \varepsilon^{-1}c(\mu_h, w_h, z_h) \quad \forall z_h \in Q_0^h.$$

By Theorem 5.6, \mathbf{M}_h is well-defined provided assumptions [B1]–[B2] hold and $h \leq h_0$. Clearly any fixed point of the mapping \mathbf{M}_h (i.e. $\mathbf{M}_h(\mu_h, w_h) = (\mu_h, w_h)$) is a solution to problem (5.8)–(5.9) and vice-versa. The goal of this section is to show that the mapping \mathbf{M}_h has a unique fixed point in a small neighborhood of $(\Pi^h \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon)$. To this end, we define the following sets:

$$(5.37) \quad \mathbb{S}_h(\rho) := \{(\mu_h, v_h) \in W_\varepsilon^h \times Q_g^h; \|\mu_h - \Pi^h \sigma^\varepsilon, v_h - \mathcal{I}^h u^\varepsilon\|_\varepsilon \leq \rho\},$$

$$(5.38) \quad \mathbb{Z}_h := \{(\mu_h, w_h) \in W_\varepsilon^h \times Q_g^h; (\mu_h, \kappa_h) + b(\kappa_h, w_h) \\ = G(\kappa_h) \quad \forall \kappa_h \in W_0^h\},$$

$$(5.39) \quad \mathbb{B}_h(\rho) := \mathbb{S}_h(\rho) \cap \mathbb{Z}_h.$$

For the continuation of the chapter, we set $\ell = \min\{s, k + 1\}$, where k is the polynomial degree of the finite element spaces W^h and Q^h , and s is defined in [B1]. The next lemma shows that the distance between the center of $\mathbb{B}_h(\rho)$ and its image under the mapping \mathbf{M}_h is small.

Lemma 5.8. *Suppose assumptions [B1]–[B4] hold. Then for $h \leq h_0$, there hold the following estimate:*

$$(5.40) \quad \left\| \left(\Pi^h \sigma^\varepsilon - M_h^{(1)}(\Pi^h \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon), \mathcal{I}^h u^\varepsilon - M_h^{(2)}(\Pi^h \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon) \right) \right\|_\varepsilon \\ \leq K_6 h^{\ell-2} \|u^\varepsilon\|_{H^\ell},$$

where

$$K_6 = CK_3 \varepsilon^{-\frac{1}{2}} (K_1^{-\frac{1}{2}} + K_0^{\frac{1}{2}} K_{R_0}).$$

PROOF. To ease notation set $\omega_h = \Pi^h \sigma^\varepsilon - M_h^{(1)}(\Pi^h \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon)$, $s_h = \mathcal{I}^h u^\varepsilon - M_h^{(2)}(\Pi^h \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon)$, $r^\varepsilon = \mathcal{I}^h u^\varepsilon - u^\varepsilon$, and $\theta^\varepsilon = \Pi^h \sigma^\varepsilon - \sigma^\varepsilon$. By the definition of \mathbf{M}_h , we have for any $(\kappa_h, z_h) \in W_0^h \times Q_0^h$

$$(\omega_h, \kappa_h) + b(\kappa_h, s_h) = (\Pi^h \sigma^\varepsilon, \kappa_h) + b(\kappa_h, \mathcal{I}^h u^\varepsilon) - G(\kappa_h), \\ b(\omega_h, z_h) - \varepsilon^{-1}d(u^\varepsilon; s_h, z_h) = b(\Pi^h \sigma^\varepsilon, z_h) - \varepsilon^{-1}c(\Pi^h \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon, z_h),$$

and therefore by (5.5)–(5.6), (5.11), and by employing the mean value theorem,

$$(5.41) \quad (\omega_h, \kappa_h) + b(\kappa_h, s_h) = (\theta^\varepsilon, \kappa_h) + b(\kappa_h, r^\varepsilon),$$

$$(5.42) \quad \begin{aligned} b(\omega_h, z_h) - \varepsilon^{-1}d(u^\varepsilon; s_h, z_h) \\ = b(\theta^\varepsilon, z_h) - \varepsilon^{-1} \left(c(\Pi^h \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon, z_h) - c(\sigma^\varepsilon, u^\varepsilon, z_h) \right) \\ = -\varepsilon^{-1} \langle F'[\xi_h, y_h](\theta^\varepsilon, r^\varepsilon), z_h \rangle, \end{aligned}$$

where $\xi_h = \Pi^h \sigma^\varepsilon - \gamma \theta^\varepsilon$ and $y_h = \mathcal{I}^h u^\varepsilon - \gamma r^\varepsilon$ for some $\gamma \in [0, 1]$.

Setting $\kappa_h = \omega_h$ and $z_h = s_h$, and subtracting (5.42) from (5.41) yield

$$\begin{aligned} (\omega_h, \omega_h) + \varepsilon^{-1}d(u^\varepsilon; s_h, s_h) &= (\theta^\varepsilon, \omega_h) + b(\omega_h, r^\varepsilon) \\ &\quad + \varepsilon^{-1} \langle F'[\xi_h, y_h](\theta^\varepsilon, r^\varepsilon), s_h \rangle. \end{aligned}$$

Consequently, by [B2]–[B4], and the inverse inequality,

$$\begin{aligned} \|(\omega_h, s_h)\|_\varepsilon^2 &\leq \|\theta^\varepsilon\|_{L^2} \|\omega_h\|_{L^2} + \|\operatorname{div}(\omega_h)\|_{L^2} \|\nabla r^\varepsilon\|_{L^2} \\ &\quad + \varepsilon^{-1} \|F'[\xi_h, y_h](\theta^\varepsilon, r^\varepsilon)\|_{H^{-1}} \|s_h\|_{H^1} + K_0 \varepsilon^{-1} \|s_h\|_{L^2}^2 \\ &\leq \|\theta^\varepsilon\|_{L^2} \|\omega_h\|_{L^2} + h^{-1} \|\omega_h\|_{L^2} \|\nabla r^\varepsilon\|_{L^2} \\ &\quad + C \varepsilon^{-1} \|(\xi_h, y_h)\|_{X \times Y} (\|\theta^\varepsilon\|_{L^2} + \|r^\varepsilon\|_{H^1}) \|s_h\|_{H^1} + K_0 \varepsilon^{-1} \|s_h\|_{L^2}^2 \\ &\leq \|\theta^\varepsilon\|_{L^2} \|\omega_h\|_{L^2} + h^{-1} \|\omega_h\|_{L^2} \|\nabla r^\varepsilon\|_{L^2} \\ &\quad + CK_3 \varepsilon^{-1} (\|\theta^\varepsilon\|_{L^2} + \|r^\varepsilon\|_{H^1}) \|s_h\|_{H^1} + K_0 \varepsilon^{-1} \|s_h\|_{L^2}^2. \end{aligned}$$

Using the Cauchy-Schwarz and inverse inequalities, and rearranging terms, give us

$$(5.43) \quad \begin{aligned} \|(\omega_h, s_h)\|_\varepsilon^2 &\leq C \left(\|\theta^\varepsilon\|_{L^2}^2 + h^{-2} \|\nabla r^\varepsilon\|_{L^2}^2 \right. \\ &\quad \left. + K_1^{-1} K_3^2 \varepsilon^{-1} (\|\theta^\varepsilon\|_{L^2}^2 + \|r^\varepsilon\|_{H^1}^2) + K_0 \varepsilon^{-1} \|s_h\|_{L^2}^2 \right) \\ &\leq C \left(h^{2\ell-4} \|\sigma^\varepsilon\|_{H^{\ell-2}}^2 + h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 + K_0 \varepsilon^{-1} \|s_h\|_{L^2}^2 \right. \\ &\quad \left. + K_1^{-1} K_3^2 \varepsilon^{-1} h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right) \\ &\leq C \left(K_1^{-1} K_3^2 \varepsilon^{-1} h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 + K_0 \varepsilon^{-1} \|s_h\|_{L^2}^2 \right). \end{aligned}$$

Next, we let $w \in Q_0 \cap H^p(\Omega)$ ($p \geq 3$) be the solution to the following auxiliary problem:

$$\begin{aligned} (G'_\varepsilon[u^\varepsilon])^*(w) &= s_h & \text{in } \Omega, \\ D^2 w \nu \cdot \nu &= 0 & \text{on } \partial\Omega, \end{aligned}$$

with

$$(5.44) \quad \|w\|_{H^p} \leq K_{R_0} \|s_h\|_{L^2}.$$

Setting $\kappa = D^2 w \in [H^{p-2}(\Omega)]^{n \times n}$, we have

$$\begin{aligned} (\kappa, \mu) + b(\mu, z) &= 0 & \forall \mu \in W_0, \\ b(\kappa, z) - \varepsilon^{-1}d^*(u^\varepsilon; w, z) &= -\varepsilon^{-1}(s_h, z) & \forall z \in Q_0. \end{aligned}$$

Thus, by (5.41)–(5.42),

$$\begin{aligned}
\varepsilon^{-1} \|s_h\|_{L^2}^2 &= -b(\kappa, s_h) + \varepsilon^{-1} d^*(u^\varepsilon; w, s_h) \\
&= -b(\Pi^h \kappa, s_h) + \varepsilon^{-1} d(u^\varepsilon; s_h, w) \\
&= (\omega_h, \Pi^h \kappa) - (\theta^\varepsilon, \Pi^h \kappa) - b(\Pi^h \kappa, r^\varepsilon) + \varepsilon^{-1} d(u^\varepsilon; s_h, w) \\
&= (\omega_h, \kappa) + (\omega_h, \Pi^h \kappa - \kappa) - (\theta^\varepsilon, \Pi^h \kappa) \\
&\quad - b(\Pi^h \kappa, r^\varepsilon) + \varepsilon^{-1} d(u^\varepsilon; s_h, w) \\
&= -b(\omega_h, w) + (\omega_h, \Pi^h \kappa - \kappa) \\
&\quad - (\theta^\varepsilon, \Pi^h \kappa) - b(\Pi^h \kappa, r^\varepsilon) + \varepsilon^{-1} d(u^\varepsilon; s_h, w) \\
&= -b(\omega_h, w - \mathcal{I}^h w) + (\omega_h, \Pi^h \kappa - \kappa) \\
&\quad - (\theta^\varepsilon, \Pi^h \kappa) - b(\Pi^h \kappa, r^\varepsilon) + \varepsilon^{-1} d(u^\varepsilon; s_h, w - \mathcal{I}^h w) \\
&\quad + \varepsilon^{-1} \langle F'[\xi_h, y_h](\theta^\varepsilon, r^\varepsilon), \mathcal{I}^h w \rangle \\
&\leq \|\operatorname{div}(\omega_h)\|_{L^2} \|\nabla(w - \mathcal{I}^h w)\|_{L^2} + \|\omega_h\|_{L^2} \|\Pi^h \kappa - \kappa\|_{L^2} \\
&\quad + \|\theta^\varepsilon\|_{L^2} \|\Pi^h \kappa\|_{L^2} + \|\operatorname{div}(\Pi^h \kappa)\|_{L^2} \|\nabla r^\varepsilon\|_{L^2} \\
&\quad + K_2 \varepsilon^{-1} \|s_h\|_{H^1} \|w - \mathcal{I}^h w\|_{H^1} \\
&\quad + K_3 \varepsilon^{-1} (\|\theta^\varepsilon\|_{L^2} + \|r^\varepsilon\|_{H^1}) \|\mathcal{I}^h w\|_{H^1} \\
&\leq C \left(h^{r-2} \|\omega_h\|_{L^2} + K_2 \varepsilon^{-1} h^{r-1} \|s_h\|_{H^1} + K_3 \varepsilon^{-1} h^{\ell-2} \|u^\varepsilon\|_{H^\ell} \right) \|w\|_{H^p}.
\end{aligned}$$

Therefore, using (5.44),

$$\|s_h\|_{L^2}^2 \leq C K_{R_0}^2 \varepsilon^2 \left(h^{2r-4} \|\omega_h\|_{L^2}^2 + K_2^2 \varepsilon^{-2} h^{2r-2} \|s_h\|_{H^1}^2 + K_3^2 \varepsilon^{-2} h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right).$$

Using this bound in (5.43), we have

$$\begin{aligned}
\|(\omega_h, s_h)\|_\varepsilon^2 &\leq C \left(K_3^2 \varepsilon^{-1} (K_1^{-1} + K_0 K_{R_0}^2) h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right. \\
&\quad \left. + K_0 K_{R_0}^2 \varepsilon (h^{2r-4} \|\omega_h\|_{L^2}^2 + K_2^2 \varepsilon^{-2} h^{2r-2} \|s_h\|_{H^1}^2) \right).
\end{aligned}$$

It then follows that for $h \leq h_0$,

$$\|(\omega_h, s_h)\|_\varepsilon \leq C K_3 \varepsilon^{-\frac{1}{2}} (K_1^{-\frac{1}{2}} + K_0^{\frac{1}{2}} K_{R_0}) h^{\ell-2} \|u^\varepsilon\|_{H^\ell}.$$

which is the inequality (5.40). The proof is complete. \square

Lemma 5.9. *Let [B1]–[B6] hold and suppose that $u^\varepsilon \in H^s(\Omega)$ ($s \geq 3$). Then there exists an $h_1 = h_1(\varepsilon) > 0$ such that for $h \leq \min\{h_0, h_1\}$, the mapping \mathbf{M}_h is a contracting mapping with a contracting factor of $\frac{1}{2}$ in the ball $\mathbb{B}_h(\rho_0)$, where*

$$\begin{aligned}
\rho_0 &:= (K_7 R(h))^{-1} \\
h_1 &:= \min \left\{ (K_7 K_G)^{-\frac{1}{\alpha}}, (K_7 R(h_1) \|u^\varepsilon\|_{H^\ell})^{\frac{1}{2-\ell}} \right\}, \\
K_7 &:= C \varepsilon^{-\frac{1}{2}} (K_1^{-\frac{1}{2}} + K_0^{\frac{1}{2}} K_{R_0}),
\end{aligned}$$

and $\alpha > 0$ is defined in [B6]. That is, for all $(\mu_h, v_h), (\kappa_h, w_h) \in \mathbb{B}_h(\rho_0)$

$$\|\mathbf{M}_h(\mu_h - \kappa_h, v_h - w_h)\|_\varepsilon \leq \frac{1}{2} \|(\mu_h - \kappa_h, v_h - w_h)\|_\varepsilon.$$

PROOF. Let $(\mu_h, v_h), (\kappa_h, w_h) \in \mathbb{B}_h(\rho_0)$, and to ease notation we set

$$M_h^{(1)} = M_h^{(1)}(\mu_h, v_h) - M_h^{(1)}(\kappa_h, w_h), \quad M_h^{(2)} = M_h^{(2)}(\mu_h, v_h) - M_h^{(2)}(\kappa_h, w_h).$$

Using the definition of \mathbf{M}_h and employing the mean value theorem, we conclude that for all $(\chi_h, z_h) \in W_0^h \times Q_0^h$,

$$(5.45) \quad (M_h^{(1)}, \chi_h) + b(\chi_h, M_h^{(2)}) = 0,$$

$$(5.46) \quad \begin{aligned} & b(M_h^{(1)}, z_h) - \varepsilon^{-1} d(u^\varepsilon; M_h^{(2)}, z_h) \\ &= \varepsilon^{-1} \left(d(u^\varepsilon; v_h - w_h, z_h) - (c(\mu_h, v_h, z_h) - c(\kappa_h, w_h, z_h)) \right) \\ &= \varepsilon^{-1} \left(d(u^\varepsilon; v_h - w_h, z_h) - \langle F'[\xi_h, y_h](\mu_h - \kappa_h, v_h - w_h), z_h \rangle \right), \end{aligned}$$

where $\xi_h = \mu_h + \gamma(\kappa_h - \mu_h)$ and $y_h = v_h + \gamma(w_h - v_h)$ for some $\gamma \in [0, 1]$. Here, we have abused the notation of ξ_h and y_h , defining them differently in two different proofs in this section.

Setting $\chi_h = M_h^{(1)}$ and $z_h = M_h^{(2)}$, subtracting (5.45) from (5.46), using assumptions [B2] and [B5], and the inverse inequality yields

$$\begin{aligned} & \left\| \left(M_h^{(1)}, M_h^{(2)} \right) \right\|_\varepsilon^2 \\ & \leq \varepsilon^{-1} \left(d(u^\varepsilon; w_h - v_h, M_h^{(2)}) - \langle F'[\xi_h, y_h](\kappa_h - \mu_h, w_h - v_h), M_h^{(2)} \rangle \right) \\ & \quad + K_0 \varepsilon^{-1} \|M_h^{(2)}\|_{L^2}^2 \\ & = \varepsilon^{-1} \left(\langle F'[\sigma^\varepsilon, u^\varepsilon](D^2 w_h - D^2 v_h, w_h - v_h) - F'[\sigma^\varepsilon, u^\varepsilon](\kappa_h - \mu_h, w_h - v_h), M_h^{(2)} \rangle \right) \\ & \quad + \varepsilon^{-1} \langle (F'[\sigma^\varepsilon, u^\varepsilon] - F'[\xi_h, y_h])(\kappa_h - \mu_h, w_h - v_h), M_h^{(2)} \rangle + K_0 \varepsilon^{-1} \|M_h^{(2)}\|_{L^2}^2 \\ & \leq \varepsilon^{-1} \left(\langle F'[\sigma^\varepsilon, u^\varepsilon](D^2(w_h - v_h) - (\kappa_h - \mu_h), 0), M_h^{(2)} \rangle \right) + K_0 \varepsilon^{-1} \|M_h^{(2)}\|_{L^2}^2 \\ & \quad + \varepsilon^{-1} R(h) (\|\sigma^\varepsilon - \xi_h\|_{L^2} + \|u^\varepsilon - y_h\|_{H^1}) \left\| (\kappa_h - \mu_h, w_h - v_h) \right\|_\varepsilon \|M_h^{(2)}\|_{H^1} \\ & \leq C \varepsilon^{-1} \left(K_G h^\alpha + R(h) (h^{\ell-2} \|u^\varepsilon\|_{H^\ell} + \rho_0) \right) \\ & \quad \times \left\| (\kappa_h - \mu_h, w_h - v_h) \right\|_\varepsilon \|M_h^{(2)}\|_{H^1} + K_0 \varepsilon^{-1} \|M_h^{(2)}\|_{L^2}^2, \end{aligned}$$

and therefore

$$(5.47) \quad \left\| \left(M_h^{(1)}, M_h^{(2)} \right) \right\|_\varepsilon^2 \leq C K_1^{-1} \varepsilon^{-1} \left(K_G^2 h^{2\alpha} + R^2(h) (h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 + \rho_0^2) \right) \times \left\| (\kappa_h - \mu_h, w_h - v_h) \right\|_\varepsilon^2 + K_0 \varepsilon^{-1} \|M_h^{(2)}\|_{L^2}^2.$$

Next, we let $z \in Q_0 \cap H^p(\Omega)$ ($p \geq 3$) be the solution to the following auxiliary problem:

$$\begin{aligned} (G'_\varepsilon[u^\varepsilon])^*(z) &= M_h^{(2)} & \text{in } \Omega, \\ D^2 z \nu \cdot \nu &= 0 & \text{on } \partial\Omega, \end{aligned}$$

with

$$\|z\|_{H^p} \leq K_{R_0} \|M_h^{(2)}\|_{L^2}.$$

Letting $\chi = D^2 z$, we have

$$(\chi, \lambda) + b(\lambda, z) = 0 \quad \forall \lambda \in W_0,$$

$$b(\chi, y) - \varepsilon^{-1} d^*(u^\varepsilon; z, y) = -\varepsilon^{-1} (M_h^{(2)}, y) \quad \forall y \in Q_0,$$

and hence by (5.45)–(5.46),

$$\begin{aligned} \varepsilon^{-1} \|M_h^{(2)}\|_{L^2}^2 &= -b(\chi, M_h^{(2)}) + \varepsilon^{-1} d^*(u^\varepsilon; z, M_h^{(2)}) \\ &= (M_h^{(1)}, \Pi^h \chi) + \varepsilon^{-1} d(u^\varepsilon; M_h^{(2)}, z) \\ &= (M_h^{(1)}, \chi) + \varepsilon^{-1} d(u^\varepsilon; M_h^{(2)}, z) + (M_h^{(1)}, \Pi^h \chi - \chi) \\ &= -b(M_h^{(1)}, z - \mathcal{I}^h z) + \varepsilon^{-1} d(u^\varepsilon; M_h^{(2)}, z - \mathcal{I}^h z) + (M_h^{(1)}, \Pi^h \chi - \chi) \\ &\quad + \varepsilon^{-1} \left(\left\langle F'[\xi_h, y_h](\kappa_h - \mu_h, w_h - v_h), \mathcal{I}^h z \right\rangle - d(u^\varepsilon; w_h - v_h, \mathcal{I}^h z) \right) \\ &\leq \|\operatorname{div}(M_h^{(1)})\|_{L^2} \|\nabla(z - \mathcal{I}^h z)\|_{L^2} + K_2 \varepsilon^{-1} \|M_h^{(2)}\|_{H^1} \|z - \mathcal{I}^h z\|_{H^1} \\ &\quad + \|M_h^{(1)}\|_{L^2} \|\Pi^h \chi - \chi\|_{L^2} + C \varepsilon^{-1} \left(K_G h^\alpha + R(h) (h^{\ell-2} \|u^\varepsilon\|_{H^\ell} + \rho_0) \right) \\ &\quad \times \|(\kappa_h - \mu_h, w_h - v_h)\|_\varepsilon \|\mathcal{I}^h z\|_{H^1} \\ &\leq C K_{R_0} \left(K_2 \varepsilon^{-1} h^{r-1} \|M_h^{(2)}\|_{H^1} + h^{r-2} \|M_h^{(1)}\|_{L^2} \right. \\ &\quad \left. + C \varepsilon^{-1} \left[K_G h^\alpha + R(h) (h^{\ell-2} \|u^\varepsilon\|_{H^\ell} + \rho_0) \right] \right) \\ &\quad \times \|(\kappa_h - \mu_h, w_h - v_h)\|_\varepsilon \|M_h^{(2)}\|_{L^2}. \end{aligned}$$

Thus,

$$\begin{aligned} \|M_h^{(2)}\|_{L^2}^2 &\leq C K_{R_0}^2 \varepsilon^2 \left(K_2^2 \varepsilon^{-2} h^{2r-2} \|M_h^{(2)}\|_{H^1}^2 + h^{2r-4} \|M_h^{(1)}\|_{L^2}^2 \right. \\ &\quad \left. + \varepsilon^{-2} \left[K_G^2 h^{2\alpha} + R^2(h) (h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 + \rho_0^2) \right] \|(\kappa_h - \mu_h, w_h - v_h)\|_\varepsilon^2 \right). \end{aligned}$$

Using the above bound in inequality (5.47) yields for $h \leq h_0$

$$\begin{aligned} &\left\| \left(M_h^{(1)}, M_h^{(2)} \right) \right\|_\varepsilon \\ &\leq C K_1^{-1} \varepsilon^{-1} \left(K_G^2 h^{2\alpha} + R^2(h) (h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 + \rho_0^2) \right) \\ &\quad \times \|(\kappa_h - \mu_h, w_h - v_h)\|_\varepsilon^2 + K_0 \varepsilon^{-1} \|M_h^{(2)}\|_{L^2}^2 \\ &\leq C \varepsilon^{-1} (K_1^{-1} + K_0 K_{R_0}^2) \left(K_G^2 h^{2\alpha} + R^2(h) (h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 + \rho_0^2) \right) \\ &\quad \times \|(\kappa_h - \mu_h, w_h - v_h)\|_\varepsilon^2. \end{aligned}$$

It then follows from the definition of ρ_0 that for $h \leq \min\{h_0, h_1\}$

$$\begin{aligned} &\left\| \left(M_h^{(1)}, M_h^{(2)} \right) \right\|_\varepsilon \\ &\leq K_6 \left(K_G h^\alpha + R(h) (h^{\ell-2} \|u^\varepsilon\|_{H^\ell} + \rho_0) \right) \|(\kappa_h - \mu_h, w_h - v_h)\|_\varepsilon \\ &\leq \frac{1}{2} \|(\kappa_h - \mu_h, w_h - v_h)\|_\varepsilon. \end{aligned}$$

□

Theorem 5.10. *Under the same assumptions of Lemma 5.9, there exists an $h_2 = h_2(\varepsilon) > 0$ such that for $h \leq \min\{h_0, h_2\}$ (5.8)–(5.9) has a locally unique solution, where h_2 is chosen such that*

$$h_2 = \min \left\{ (K_7 K_G)^{-\frac{1}{\alpha}}, \left(2K_6 K_7 R(h_2) \|u^\varepsilon\|_{H^\ell} \right)^{\frac{1}{2-\ell}} \right\}.$$

Furthermore, there holds the following error estimate:

$$(5.48) \quad \|(\sigma^\varepsilon - \sigma_h^\varepsilon, u^\varepsilon - u_h^\varepsilon)\|_\varepsilon \leq h^{\ell-2} K_8 \|u^\varepsilon\|_{H^\ell},$$

where

$$K_8 = CK_6 = CK_3 \varepsilon^{-\frac{1}{2}} (K_1^{-\frac{1}{2}} + K_0^{\frac{1}{2}} K_{R_0}).$$

PROOF. Let

$$\rho_1 = 2K_6 h^{\ell-2} \|u^\varepsilon\|_{H^\ell}.$$

Then for $h \leq \min\{h_0, h_2\}$, there holds $\rho_1 \leq \rho_0$.

Thus noting $h_2 \leq h_1$, for any $(\mu_h, v_h) \in \mathbb{B}_h(\rho_1)$, we use Lemmas 5.8 and 5.9 to conclude that

$$\begin{aligned} & \left\| \left(\Pi^h \sigma^\varepsilon - M_h^{(1)}(\mu_h, v_h), \mathcal{I}^h u^\varepsilon - M_h^{(2)}(\mu_h, v_h) \right) \right\|_\varepsilon \\ & \leq \left\| \left(\Pi^h \sigma^\varepsilon - M_h^{(1)}(\Pi^h \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon), \mathcal{I}^h u^\varepsilon - M_h^{(2)}(\Pi^h \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon) \right) \right\|_\varepsilon \\ & \quad + \left\| \left(M_h^{(1)}(\Pi^h \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon) - M_h^{(1)}(\mu_h, v_h), M_h^{(2)}(\Pi^h \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon) - M_h^{(2)}(\mu_h, v_h) \right) \right\|_\varepsilon \\ & \leq K_6 h^{\ell-2} \|u^\varepsilon\|_{H^\ell} + \frac{1}{2} \left\| (\Pi^h \sigma^\varepsilon - \mu_h, \mathcal{I}^h u^\varepsilon - v_h) \right\|_\varepsilon \\ & \leq \frac{\rho_1}{2} + \frac{\rho_1}{2} = \rho_1, \end{aligned}$$

and so $\mathbf{M}_h(\mu_h, v_h) \in \mathbb{B}_h(\rho_1)$. It is clear that \mathbf{M}_h is a continuous mapping. It follows from Banach's Fixed Point Theorem [42] that \mathbf{M}_h has a unique fixed point $(\sigma_h^\varepsilon, u_h^\varepsilon)$ in the ball $\mathbb{B}_h(\rho_1)$, which is the unique solution to (5.8)–(5.9).

To obtain the error estimate (5.48), we use the triangle inequality to conclude

$$\begin{aligned} & \|(\sigma^\varepsilon - \sigma_h^\varepsilon, u^\varepsilon - u_h^\varepsilon)\|_\varepsilon \\ & \leq \|(\sigma^\varepsilon - \Pi^h \sigma^\varepsilon, u^\varepsilon - \mathcal{I}^h u^\varepsilon)\|_\varepsilon + \|(\Pi^h \sigma^\varepsilon - \sigma_h^\varepsilon, \mathcal{I}^h u^\varepsilon - u_h^\varepsilon)\|_\varepsilon \\ & \leq Ch^{\ell-2} \|u^\varepsilon\|_{H^\ell} + C\rho_1 \leq CK_6 h^{\ell-2} \|u^\varepsilon\|_{H^\ell}. \end{aligned}$$

□

Note that the error estimates of $\|u^\varepsilon - u_h^\varepsilon\|_{H^1}$ in Theorem 5.10 are sub-optimal. In the next theorem, we employ a duality argument to improve the above error estimates and to also obtain L^2 error estimates.

Theorem 5.11. *In addition to the hypotheses of Theorem 5.10, suppose that $p \geq 4$ in assumption [B2]. Then there hold the following error estimates:*

$$\begin{aligned} \|u^\varepsilon - u_h^\varepsilon\|_{L^2} & \leq K_{R_0} \left(K_9 h^{\ell-2+\min\{2,\alpha\}} \|u^\varepsilon\|_{H^\ell} + K_8^2 R(h) h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right), \\ \|u^\varepsilon - u_h^\varepsilon\|_{H^1} & \leq K_{R_1} \left(K_9 h^{\ell-2+\min\{1,\alpha\}} \|u^\varepsilon\|_{H^\ell} + K_8^2 R(h) h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right), \end{aligned}$$

where

$$K_9 = CK_8 \max\{K_2, K_G\}.$$

PROOF. To ease notation, we set

$$\pi^\varepsilon := \sigma^\varepsilon - \sigma_h^\varepsilon, \quad e^\varepsilon := u^\varepsilon - u_h^\varepsilon.$$

We note that by using the mean value theorem, there hold the following error equations:

$$(5.49) \quad (\pi^\varepsilon, \mu_h) + b(\mu_h, e^\varepsilon) = 0 \quad \forall \mu_h \in W_0^h,$$

$$(5.50) \quad b(\pi^\varepsilon, v_h) - \langle F'[\xi_h, y_h](\pi^\varepsilon, e^\varepsilon), v_h \rangle = 0 \quad \forall v_h \in Q_0^h,$$

where $\xi_h = \sigma^\varepsilon - \gamma\pi^\varepsilon$, $y_h = u^\varepsilon - \gamma e^\varepsilon$ for some $\gamma \in [0, 1]$. Again, we have abused the notation of ξ_h and y_h , defining them differently in two separate proofs.

Next, let $w_m \in H^{p-m}(\Omega) \cap Q_0$ ($m = 0, 1; p \geq 4$) be the solution to the following auxiliary problem:

$$\begin{aligned} (G'_\varepsilon[u^\varepsilon])^*(w_m) &= (-1)^m \Delta^m e^\varepsilon && \text{in } \Omega, \\ D^2 w_m \nu \cdot \nu &= 0 && \text{on } \partial\Omega, \end{aligned}$$

with

$$(5.51) \quad \|w_m\|_{H^{p-m}} \leq K_{R_m} \|\nabla^m e^\varepsilon\|_{L^2}.$$

Here, we have used the notation $\Delta^1 = \Delta$, $\nabla^1 = \nabla$, and Δ^0, ∇^0 are the identity operators on Q . Setting $\kappa_m = D^2 w_m \in [H^{p-m-2}(\Omega)]^{n \times n}$, we then have

$$\begin{aligned} (\kappa_m, \mu) + b(\mu, w_m) &= 0 && \forall \mu \in W_0, \\ b(\kappa_m, v) - \varepsilon^{-1} d^*(u^\varepsilon; w_m, v) &= -\varepsilon^{-1} (\nabla^m e^\varepsilon, \nabla^m v) && \forall v \in Q_0. \end{aligned}$$

Therefore,

$$\begin{aligned} \varepsilon^{-1} \|\nabla^m e^\varepsilon\|_{L^2}^2 &= -b(\kappa_m, e^\varepsilon) + \varepsilon^{-1} d^*(u^\varepsilon; w_m, e^\varepsilon) \\ &= (\pi_h^\varepsilon, \Pi^h \kappa_m) + \varepsilon^{-1} d(u^\varepsilon; e^\varepsilon, w_m) - b(\kappa_m - \Pi^h \kappa_m, e^\varepsilon) \\ &= (\pi^\varepsilon, \kappa_m) + \varepsilon^{-1} d(u^\varepsilon; e^\varepsilon, w_m) \\ &\quad - b(\kappa_m - \Pi^h \kappa_m, u^\varepsilon - \mathcal{I}^h u^\varepsilon) + (\pi^\varepsilon, \Pi^h \kappa_m - \kappa_m) \\ &= -b(\pi^\varepsilon, w_m) + \varepsilon^{-1} d(u^\varepsilon; e^\varepsilon, w_m) \\ &\quad - b(\kappa_m - \Pi^h \kappa_m, u^\varepsilon - \mathcal{I}^h u^\varepsilon) + (\pi^\varepsilon, \Pi^h \kappa_m - \kappa_m) \\ &= -b(\pi^\varepsilon, w_m - \mathcal{I}^h w_m) + \varepsilon^{-1} d(u^\varepsilon; e^\varepsilon, w_m - \mathcal{I}^h w_m) \\ &\quad - b(\kappa_m - \Pi^h \kappa_m, u^\varepsilon - \mathcal{I}^h u^\varepsilon) + (\pi^\varepsilon, \Pi^h \kappa_m - \kappa_m) \\ &\quad + \varepsilon^{-1} d(u^\varepsilon; e^\varepsilon, \mathcal{I}^h w_m) - \varepsilon^{-1} \langle F'[\xi_h, y_h](\pi^\varepsilon, e^\varepsilon), \mathcal{I}^h w_m \rangle \\ &= -b(\pi^\varepsilon, w_m - \mathcal{I}^h w_m) + \varepsilon^{-1} d(u^\varepsilon; e^\varepsilon, w_m - \mathcal{I}^h w_m) \\ &\quad - b(\kappa_m - \Pi^h \kappa_m, u^\varepsilon - \mathcal{I}^h u^\varepsilon) + (\pi^\varepsilon, \Pi^h \kappa_m - \kappa_m) \\ &\quad + \varepsilon^{-1} \langle F'[\sigma^\varepsilon, u^\varepsilon](D^2 e^\varepsilon - \pi^\varepsilon, 0), \mathcal{I}^h w_m \rangle \\ &\quad + \varepsilon^{-1} \langle (F'[\sigma^\varepsilon, u^\varepsilon] - F'[\xi_h, y_h])(\pi^\varepsilon, e^\varepsilon), \mathcal{I}^h w_m \rangle. \end{aligned}$$

Bounding the right-hand side in the last expression, we have

$$\begin{aligned}
& \varepsilon^{-1} \|\nabla^m e^\varepsilon\|_{L^2}^2 \\
& \leq \|\operatorname{div}(\pi^\varepsilon)\|_{L^2} \|\nabla(w_m - \mathcal{I}^h w_m)\|_{L^2} + K_2 \varepsilon^{-1} \|e^\varepsilon\|_{H^1} \|w_m - \mathcal{I}^h w_m\|_{H^1} \\
& \quad + \|\operatorname{div}(\kappa_m - \Pi^h \kappa_m)\|_{L^2} \|\nabla(u^\varepsilon - \mathcal{I}^h u^\varepsilon)\|_{L^2} + \|\pi^\varepsilon\|_{L^2} \|\Pi^h \kappa_m - \kappa_m\|_{L^2} \\
& \quad + \varepsilon^{-1} \left\langle F'[u^\varepsilon, \sigma^\varepsilon](D^2 e^\varepsilon - \pi^\varepsilon, 0), \mathcal{I}^h w_m \right\rangle \\
& \quad + \varepsilon^{-1} \left\langle (F'[\xi_h, y_h] - F'[\sigma^\varepsilon, u^\varepsilon])(\pi^\varepsilon, e^\varepsilon), \mathcal{I}^h w_m \right\rangle \\
& \leq C \left(h^{3-m} \|\pi^\varepsilon\|_{H^1} + K_2 \varepsilon^{-1} h^{2-m} \|e^\varepsilon\|_{H^1} + \|\nabla(u^\varepsilon - \mathcal{I}^h u^\varepsilon)\|_{L^2} \right. \\
& \quad + h \|\pi^\varepsilon\|_{L^2} + \varepsilon^{-1} K_G h^\alpha \|\pi^\varepsilon, e^\varepsilon\|_\varepsilon \\
& \quad \left. + \varepsilon^{-1} R(h) (\|\xi_h - \sigma^\varepsilon\|_{L^2} + \|y_h - u^\varepsilon\|_{H^1}) \|\pi^\varepsilon, e^\varepsilon\|_\varepsilon \right) \|w_m\|_{H^{p-m}} \\
& \leq C K_{R_m} \left(h^{3-m} \|\pi^\varepsilon\|_{H^1} + K_2 \varepsilon^{-1} h^{2-m} \|e^\varepsilon\|_{H^1} \right. \\
& \quad + h^{\ell-1} \|u^\varepsilon\|_{H^\ell} + h \|\pi^\varepsilon\|_{L^2} + \varepsilon^{-1} K_G h^\alpha \|\pi^\varepsilon, e^\varepsilon\|_\varepsilon \\
& \quad \left. + \varepsilon^{-1} R(h) \|\pi^\varepsilon, e^\varepsilon\|_\varepsilon^2 \right) \|\nabla^m e^\varepsilon\|_{L^2} \\
& \leq C K_{R_m} \varepsilon^{-1} \left((K_2 h^{2-m} + K_G h^\alpha) \|\pi^\varepsilon, e^\varepsilon\|_\varepsilon + R(h) \|\pi^\varepsilon, e^\varepsilon\|_\varepsilon^2 \right) \|\nabla^m e^\varepsilon\|_{L^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\nabla^m e^\varepsilon\|_{L^2} & \leq C K_{R_m} \left((K_2 h^{2-m} + K_G h^\alpha) \|\pi^\varepsilon, e^\varepsilon\|_\varepsilon + R(h) \|\pi^\varepsilon, e^\varepsilon\|_\varepsilon^2 \right) \\
& \leq C K_8 K_{R_m} \left((K_2 h^{2-m} + K_G h^\alpha) h^{\ell-2} \|u^\varepsilon\|_{H^\ell} + R(h) h^{2\ell-4} K_8 \|u^\varepsilon\|_{H^\ell}^2 \right).
\end{aligned}$$

The proof is complete. \square

5.4. Generalizations: the case of degenerate equations

In this section, we generalize the analysis of the preceding sections to handle cases in which condition [B2] fails to hold, namely when the inequality

$$(5.52) \quad \langle F'[\sigma^\varepsilon, u^\varepsilon](\chi, v), v \rangle \geq K_1 \|v\|_{H^1}^2 - K_0 \|v\|_{L^2}^2 \quad \forall v \in Q_0$$

does not hold for any positive constant K_1 . Thus, in this section we consider cases in which the operator F may become degenerate (i.e. has vanishing smallest eigenvalue) at the solution u^ε . An instance of such a case arises when studying mixed finite element approximations of the infinity-Laplacian equation (cf. Section 6.3).

Here, we introduce a more flexible mixed finite element formulation to overcome this difficulty. To this end, we rewrite (2.9)–(2.11)₃ into the following system of second order equations:

$$(5.53) \quad \tilde{\sigma}^\varepsilon - D^2 u^\varepsilon - \tau I_{n \times n} u^\varepsilon = 0,$$

$$(5.54) \quad \varepsilon \operatorname{div}(\operatorname{div}(\tilde{\sigma}^\varepsilon)) + \varepsilon \tau \operatorname{tr}(\tilde{\sigma}^\varepsilon) + \tilde{F}(\tilde{\sigma}^\varepsilon, u^\varepsilon) = 0,$$

where

$$\tilde{F}(\tilde{\sigma}^\varepsilon, u^\varepsilon) := -2\varepsilon \tau \Delta u^\varepsilon - n \varepsilon \tau^2 u^\varepsilon + F(\tilde{\sigma}^\varepsilon - \tau I_{n \times n} u^\varepsilon, u^\varepsilon),$$

$I_{n \times n}$ denotes the $n \times n$ identity matrix and τ is a nonnegative constant that is independent of ε . Clearly, (5.53)–(5.54) is the same as (5.1)–(5.2) with $\tilde{\sigma}^\varepsilon = \sigma^\varepsilon + \tau I_{n \times n} u^\varepsilon$.

The variational formulation of (5.53)–(5.54) is then defined as seeking $(\tilde{\sigma}^\varepsilon, u^\varepsilon) \in \widetilde{W}_\varepsilon \times Q_g$ such that

$$(5.55) \quad (\tilde{\sigma}^\varepsilon, \mu) + \tilde{b}(\mu, u^\varepsilon) = G(\mu) \quad \forall \mu \in W_0,$$

$$(5.56) \quad \tilde{b}(\tilde{\sigma}^\varepsilon, v) - \varepsilon^{-1} \tilde{c}(\tilde{\sigma}^\varepsilon, u^\varepsilon, v) = 0 \quad \forall v \in Q_0,$$

where

$$\widetilde{W}_\varepsilon := \{\mu \in W; \mu \nu \cdot \nu|_{\partial\Omega} = \varepsilon + \tau g\},$$

$$\tilde{b}(\mu, v) := (\operatorname{div}(\mu), \nabla v) - \tau(\operatorname{tr}(\mu), v),$$

$$\tilde{c}(\mu, v, w) := \langle \tilde{F}(\mu, v), w \rangle = 2\varepsilon\tau(\nabla v, \nabla w) - \varepsilon n\tau^2(v, w) + (F(\mu - \tau I_{n \times n} v, v), w),$$

and $G(\mu)$ is defined by (5.7). We note that (5.55)–(5.56) is the same as (5.5)–(5.6) for the case $\tau = 0$.

Based on the variational formulation (5.53)–(5.54), we define our mixed finite element method of (2.9)–(2.11)₃ as seeking $(\tilde{\sigma}_h^\varepsilon, u_h^\varepsilon) \in \widetilde{W}_\varepsilon^h \times Q_g^h$ (where $\widetilde{W}_\varepsilon^h := W^h \cap \widetilde{W}_\varepsilon$) such that

$$(5.57) \quad (\tilde{\sigma}_h^\varepsilon, \mu_h) + \tilde{b}(\mu_h, u_h^\varepsilon) = G(\mu_h) \quad \forall \mu_h \in W_0,$$

$$(5.58) \quad \tilde{b}(\tilde{\sigma}_h^\varepsilon, v_h) - \varepsilon^{-1} \tilde{c}(\tilde{\sigma}_h^\varepsilon, u_h^\varepsilon, v_h) = 0 \quad \forall v_h \in Q_0.$$

The specific goal of this section is to analyze the finite element method (5.57)–(5.58) and to determine what conditions are sufficient to show existence, uniqueness, and error estimates of the solution. Clearly, the finite element method and (5.8)–(5.9) have a similar structure, and therefore, one would expect that most of the analysis in the previous sections can be inherited in the present case. However, one issue of concern is that we have changed the bilinear form $b(\cdot, \cdot)$ in the new formulation, leading to question whether the inf-sup condition (cf. Lemma 5.1) still holds. As is now well-known, this is a crucial ingredient in mixed finite element analysis, and we have used it copiously in the analysis above (albeit, indirectly). We appease these worries in the next lemma, showing that the inf-sup condition still holds provided τ is small enough. The reason for using the new bilinear form $\tilde{b}(\cdot, \cdot)$ will become clear later (see (5.68)).

Lemma 5.12. *There exists positive constants τ_0, C depending only on n and Ω such that for $\tau \leq \tau_0$ there holds the following inequality for any $v_h \in Q_0^h$:*

$$(5.59) \quad \sup_{\mu_h \in W_0^h} \frac{\tilde{b}(\mu_h, v_h)}{\|\mu_h\|_{H^1}} \geq C \|v_h\|_{H^1}.$$

PROOF. By Poincaré's inequality there exists a positive constant C_p that depends only on Ω and n such that for all $v \in H_0^1(\Omega)$

$$\|v\|_{L^2} \leq C_p \|\nabla v\|_{L^2}.$$

For $v_h \in Q_0^h \subset H_0^1(\Omega)$, set $\kappa_h = I_{n \times n} v_h \in W_0^h$. Then

$$\begin{aligned} \sup_{\mu_h \in W_0^h} \frac{\tilde{b}(\mu_h, v_h)}{\|\mu_h\|_{H^1}} &\geq \frac{\tilde{b}(\kappa_h, v_h)}{\|\kappa_h\|_{H^1}} = \frac{(\operatorname{div}(\kappa_h), \nabla v_h) - \tau(\operatorname{tr}(\kappa_h), v_h)}{\sqrt{n}\|v_h\|_{H^1}} \\ &= \frac{\|\nabla v_h\|_{L^2}^2 - n\tau\|v_h\|_{L^2}^2}{\sqrt{n}\|v_h\|_{H^1}} \geq \frac{(1 - C_p^2 n\tau)\|\nabla v_h\|_{L^2}^2}{\sqrt{n}\|v_h\|_{H^1}} \\ &\geq \frac{1}{2\sqrt{n}} \min\{(1 - C_p^2 n\tau), C_p\} \|v_h\|_{H^1}. \end{aligned}$$

Choosing $\tau_0 = \frac{1}{2}C_p^{-2}n$, we obtain the desired inequality (5.59). \square

Next, we introduce the analogous linearization problem and mixed formulation to (5.53)–(5.54). That is, instead of (5.20)–(5.21), we write

$$(5.60) \quad \tilde{\chi} - D^2 v - \tau I_{n \times n} v = 0 \quad \text{in } \Omega,$$

$$(5.61) \quad \varepsilon \operatorname{div}(\operatorname{div}(\tilde{\chi})) + \varepsilon \tau \operatorname{tr}(\tilde{\chi}) + \tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon](D^2 v, v) = \varphi \quad \text{in } \Omega,$$

$$(5.62) \quad \tilde{\chi} \nu \cdot \nu = 0, \quad v = 0 \quad \text{on } \partial\Omega,$$

where we define

$$\tilde{F}'[\omega, y](\mu, w) := -2\varepsilon\tau\Delta w - \varepsilon n\tau^2 w + F'[\omega - \tau I_{n \times n} y, y](\mu, w),$$

and $F'\cdot, \cdot$ is defined by (1.13). We note that (recall $\sigma^\varepsilon = D^2 u^\varepsilon$)

$$\tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon](\mu, w) = -2\varepsilon\tau\Delta w - \varepsilon n\tau^2 w + F'[\sigma^\varepsilon, u^\varepsilon](\mu, w).$$

The variational formulation of (5.60)–(5.61) is then defined as seeking $(\tilde{\chi}, v) \in W_0 \times Q_0$ such that

$$(\tilde{\chi}, \mu) + \tilde{b}(\mu, v) = 0 \quad \forall \mu \in W_0,$$

$$\tilde{b}(\tilde{\chi}, w) - \varepsilon^{-1} \tilde{d}(u^\varepsilon; v, w) = -\varepsilon^{-1} \langle \varphi, w \rangle \quad \forall w \in Q_0,$$

where

$$\begin{aligned} \tilde{d}(u^\varepsilon; v, w) &:= \left\langle \tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon](D^2 v, v), w \right\rangle \\ &= 2\varepsilon\tau(\nabla v, \nabla w) - \varepsilon n\tau^2(v, w) + \langle F'[\tilde{\sigma}^\varepsilon - I_{n \times n} u^\varepsilon, u^\varepsilon](D^2 v, v), w \rangle. \end{aligned}$$

It then follows that the corresponding finite element method for the linearized problem is to find $(\tilde{\chi}_h, v_h) \in W_0^h \times Q_0^h$ such that

$$(5.63) \quad (\tilde{\chi}_h, \mu_h) + \tilde{b}(\mu_h, v_h) = 0 \quad \forall \mu_h \in W_0^h,$$

$$(5.64) \quad \tilde{b}(\tilde{\chi}_h, w_h) - \varepsilon^{-1} \tilde{d}(u^\varepsilon; v_h, w_h) = -\varepsilon^{-1} \langle \varphi, w_h \rangle \quad \forall w_h \in Q_0.$$

We now address what conditions are sufficient to show that the finite element methods (5.57)–(5.58) and (5.63)–(5.64) are well-posed. As it turns out, we are able to obtain results with weaker conditions than imposed in the previous section. Specifically, we are able to replace assumption [B2] by the following less-strict condition.

[B2] The operator $(G'_\varepsilon[u^\varepsilon])^*$ (the adjoint of $G'_\varepsilon[u^\varepsilon]$ defined in Chapter 4) is an isomorphism from $H^2(\Omega) \cap H_0^1(\Omega)$ to $(H^2(\Omega) \cap H_0^1(\Omega))^*$. That is for all $\varphi \in (H^2(\Omega) \cap H_0^1(\Omega))^*$, there exists $v \in H^2(\Omega) \cap H_0^1(\Omega)$ such that

$$\langle (G'_\varepsilon[u^\varepsilon])^*(v), w \rangle = \langle \varphi, w \rangle \quad \forall w \in H^2(\Omega) \cap H_0^1(\Omega).$$

Furthermore, there exists a positive constant $K_0 = K_0(\varepsilon)$ such that the following inequality holds:

$$(5.65) \quad \langle F'[\sigma^\varepsilon, u^\varepsilon](D^2v, v), v \rangle \geq -K_0 \|v\|_{L^2}^2.$$

and there exists $K_2 > 0$ such that

$$\|F'[\sigma^\varepsilon, u^\varepsilon]\|_{QQ^*} \leq K_2.$$

Moreover, there exists $p \geq 3$ and $K_{R_0} > 0$, $K_{R_1} > 0$ such that if $\varphi \in H^{-m}(\Omega)$ ($m = 0, 1$) and $v \in V_0$ satisfies (5.13), then $v \in H^{p-m}(\Omega)$ and

$$\|v\|_{H^{p-m}} \leq K_{R_m} \|\varphi\|_{H^{-m}}.$$

Remark 5.13. We note that the only difference between [B2] and $[\widetilde{B2}]$ are the inequalities (5.65) and (5.14). Clearly if (5.14) holds, then (5.65) holds as well, but not vice-versa.

We now address the well-posedness of the finite element method for the linearized problem (5.63)–(5.64).

Theorem 5.14. *Suppose assumptions [B1] and $[\widetilde{B2}]$ hold, $\tau \in (0, \tau_0)$, $v \in H^s(\Omega)$ ($s \geq 3$) is the unique solution to (5.17)–(5.19) and $\tilde{\chi} = D^2v + \tau I_{n \times n} v$. Then there exists an $\tilde{h}_0 = \tilde{h}_0(\varepsilon) > 0$ such that for $h \leq \tilde{h}_0$, there exists a unique solution $(\tilde{\chi}_h, w_h) \in W_0^h \times Q_0^h$ to problem (5.63)–(5.64), where*

$$\tilde{h}_0 = O\left(\min\left\{(K_0 K_2^2 K_{R_1}^2 \varepsilon^{-1} \tau^{-1})^{\frac{1}{2-2r}}, (K_0 K_{R_1}^2 \varepsilon)^{\frac{1}{4-2r}}\right\}\right), \quad r = \min\{p, k+1\}.$$

Here, k is the degree of the polynomial space of Q^h and W^h , and p is defined in [B2]. Furthermore, there hold the following error estimates:

$$(5.66) \quad \|(\chi - \chi_h, v - v_h)\|_\varepsilon \leq Ch^{\ell-2} (\tilde{K}_4 h + 1) \|u^\varepsilon\|_{H^\ell},$$

$$(5.67) \quad \|v - v_h\|_{L^2} \leq \tilde{K}_5 h^{\ell+r-4} (\tilde{K}_4 h + 1) \|u^\varepsilon\|_{H^\ell},$$

where

$$\tilde{K}_4 = O\left(\max\{K_2 \varepsilon^{-1} \tau^{-\frac{1}{2}}, K_0^{\frac{1}{2}} K_{R_1} \varepsilon^{\frac{1}{2}}\}\right), \quad \tilde{K}_5 = O\left(K_2 K_{R_1} \tau^{-\frac{1}{2}}\right),$$

$$\ell = \min\{s, k+1\},$$

and

$$\|(\mu, v)\|_\varepsilon := \|\mu\|_{L^2} + \tau^{\frac{1}{2}} \|v\|_{H^1},$$

$$\|(\mu, v)\|_\varepsilon := h \|\mu\|_{H^1} + \|(\mu, v)\|_\varepsilon.$$

PROOF. It is clear from the proof of Theorem 5.6 that we only need to verify that condition [B2] holds, but with $F'[\sigma^\varepsilon, u^\varepsilon]$ replaced by $\tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon]$.

By the definition of $\tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon]$, we have

$$\langle \tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon](D^2v, v), v \rangle = 2\varepsilon\tau \|\nabla v\|_{L^2}^2 - \varepsilon n \tau^2 \|v\|_{L^2}^2 + \langle F'[\sigma^\varepsilon, u^\varepsilon](D^2v, v), v \rangle.$$

Thus, if $[\widetilde{B2}]$ holds, then

$$(5.68) \quad \langle \tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon](D^2v, v), v \rangle \geq \tilde{K}_1 \|v\|_{H^1}^2 - \tilde{K}_0 \|v\|_{L^2}^2,$$

with

$$\tilde{K}_0 := \varepsilon n \tau^2 + K_0, \quad \tilde{K}_1 := 2\varepsilon\tau.$$

We also notice that

$$\begin{aligned}
\|\tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon]\|_{QQ^*} &= \sup_{v \in Q_0} \sup_{w \in Q_0} \frac{\langle \tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon](D^2v, v), w \rangle}{\|v\|_{H^1} \|w\|_{H^1}} \\
&= \sup_{v \in Q_0} \sup_{w \in Q_0} \frac{2\varepsilon\tau(\nabla v, \nabla w) - \varepsilon n\tau^2(v, w) + \langle F'[\sigma^\varepsilon, u^\varepsilon](D^2v, v), w \rangle}{\|v\|_{H^1} \|w\|_{H^1}} \\
&\leq \varepsilon\tau(2 + n\tau) + \|F'[\sigma^\varepsilon, u^\varepsilon](D^2v, v)\|_{QQ^*} \\
&\leq \varepsilon\tau(2 + n\tau) + K_2 =: \tilde{K}_2.
\end{aligned}$$

It then follows that [B2] holds but with $F'[\sigma^\varepsilon, u^\varepsilon]$ replaced by $\tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon]$, and the assertions of the theorem immediately follow. \square

With the well-posedness results for the linear problem established, we can now state and prove the main result of this section (compare to Theorem 5.11).

Theorem 5.15. *Suppose assumptions [B1], [B2], [B3]–[B6] hold, $u^\varepsilon \in H^3(\Omega)$ ($s \geq 3$), $R(h) = o(h^{2-\ell})$, $\tau \in (0, \tau_0)$, and there exists $\tilde{K}_3 = \tilde{K}_3(\varepsilon)$ such that (5.73) holds. Then there exists $\tilde{h}_1 = \tilde{h}_1(\varepsilon) > 0$ such that for $h \leq \min\{\tilde{h}_0, \tilde{h}_1\}$, there hold the following error estimates:*

$$(5.69) \quad \|(\tilde{\sigma}^\varepsilon - \tilde{\sigma}_h^\varepsilon, u^\varepsilon - u_h^\varepsilon)\|_\varepsilon \leq \tilde{K}_8 h^{\ell-2} \|u^\varepsilon\|_{H^\ell},$$

$$(5.70) \quad \|u^\varepsilon - u_h^\varepsilon\|_{L^2} \leq K_{R_0} \left(\tilde{K}_9 h^{\ell-2+\min\{2, \alpha\}} \|u^\varepsilon\|_{H^\ell} + \tilde{K}_8^2 R(h) h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right),$$

$$(5.71) \quad \|u^\varepsilon - u_h^\varepsilon\|_{H^1} \leq K_{R_1} \left(\tilde{K}_9 h^{\ell-2+\min\{1, \alpha\}} \|u^\varepsilon\|_{H^\ell} + \tilde{K}_8^2 R(h) h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right),$$

where

$$\tilde{K}_8 = C\varepsilon^{-\frac{1}{2}} = C\tilde{K}_3\varepsilon^{-\frac{1}{2}} \left(\tau^{-\frac{1}{2}} + K_0^{\frac{1}{2}} K_{R_0} \right),$$

$$\tilde{K}_9 = C\tilde{K}_8 \max\{K_2, K_G\},$$

$$\ell = \min\{s, k+1\},$$

and s is defined in [B1].

PROOF. The idea of the proof is to show that [B2]–[B6] hold for \tilde{F}' (and $\tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon]$) if [B2], [B3]–[B6] hold for F' (and $F'[\sigma^\varepsilon, u^\varepsilon]$). The result then follows using the same techniques as those employed in the proof of Theorem 5.11.

First, from the proof of Theorem 5.14, we know that [B2] holds for $\tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon]$. Next, if assumption [B3] holds then

$$\begin{aligned}
&\|\tilde{F}'[\omega, y](\chi, v)\|_{H^{-1}} \\
&= \sup_{z \in Q_0} \frac{2\varepsilon\tau(\nabla v, \nabla z) - \varepsilon n\tau^2(v, z) + \langle F'[\omega - \tau I_{n \times n} y, y](\chi, v), z \rangle}{\|z\|_{H^1}} \\
&\leq \varepsilon\tau(2 + n\tau) \|v\|_{H^1} + C \|(\omega - \tau I_{n \times n} y, y)\|_{X \times Y} (\|\chi\|_{L^2} + \|v\|_{H^1}).
\end{aligned}$$

If we define

$$(5.72) \quad \|(\omega, y)\|_{\tilde{X} \times \tilde{Y}} := \|(\omega - \tau I_{n \times n} y, y)\|_{X \times Y},$$

then

$$\|\tilde{F}'[\omega, y](\chi, v)\|_{H^{-1}} \leq \varepsilon\tau(2 + n\tau) \|v\|_{H^1} + C \|(\omega, y)\|_{\tilde{X} \times \tilde{Y}} (\|\chi\|_{L^2} + \|v\|_{H^1}).$$

From the definitions of Q^h and W^h , $\|(\cdot, \cdot)\|_{\tilde{X} \times \tilde{Y}}$ is well-defined on $W^h \times Q^h$ and if

$$(5.73) \quad \|(\Pi^h \sigma^\varepsilon - \gamma \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon - \gamma u^\varepsilon)\|_{\tilde{X} \times \tilde{Y}} \leq \tilde{K}_3 \quad \forall \gamma \in [0, 1],$$

then it follows that conditions [B3]–[B4] hold for \tilde{F}' with $\|(\cdot, \cdot)\|_{\tilde{X} \times \tilde{Y}}$ in place of $\|(\cdot, \cdot)\|_{X \times Y}$.

Next, for any $(\mu_h, v_h) \in \tilde{W}_\varepsilon^h \times Q_g^h$, $(\kappa_h, z_h) \in W^h \times Q^h$, and $w_h \in Q_0^h$

$$\begin{aligned} & \left\langle (\tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon] - \tilde{F}'[\mu_h, v_h])(\kappa_h, z_h), w_h \right\rangle \\ &= \left\langle (F'[\sigma^\varepsilon, u^\varepsilon] - F'[\mu_h - \tau I_{n \times n} v_h, v_h])(\kappa_h, z_h), w_h \right\rangle. \end{aligned}$$

Define $\mu_\tau \in W_\varepsilon^h$ such that

$$\mu_\tau := \mu_h - \tau I_{n \times n} v_h,$$

and notice that if

$$\|(\mathcal{I}^h \tilde{\sigma}^\varepsilon - \mu_h, \mathcal{I}^h u^\varepsilon - v_h)\|_\varepsilon \leq \delta,$$

then

$$\|(\Pi^h \sigma^\varepsilon - \mu_\tau, \mathcal{I}^h u^\varepsilon - v_h)\|_\varepsilon \leq C\delta.$$

Therefore, redefining δ if necessary, we have

$$\begin{aligned} & \left\langle (\tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon] - \tilde{F}'[\mu_h, v_h])(\kappa_h, z_h), w_h \right\rangle \\ &= \left\langle (F'[\sigma^\varepsilon, u^\varepsilon] - F'[\mu_\tau, v_h])(\kappa_h, z_h), w_h \right\rangle \\ &\leq R(h)(\|\sigma^\varepsilon - \mu_\tau\|_{L^2} + \|u^\varepsilon - v_h\|_{H^1}) \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1} \\ &\leq CR(h)(\|\tilde{\sigma}^\varepsilon - \mu_h\|_{L^2} + \|u^\varepsilon - v_h\|_{H^1}) \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1}. \end{aligned}$$

Hence, [B5] holds for $\tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon]$.

Finally, we show that condition [B6] holds for $\tilde{F}'[\tilde{\sigma}^\varepsilon, u^\varepsilon]$. Suppose that

$$(5.74) \quad (\chi_h, \kappa_h) + \tilde{b}(\kappa_h, v_h) = 0 \quad \forall \kappa_h \in W_0^h,$$

where $(\chi_h, v_h) \in W_0^h \times Q_0^h$. It then follows that

$$(\chi_h - \tau I_{n \times n} v_h, \kappa_h) + b(\kappa_h, v_h) = 0,$$

that is $(\chi_h - \tau I_{n \times n} v_h, v_h) \in \mathbb{T}_h$, where \mathbb{T}_h is defined in [B6]. Thus, if [B6] holds (with $K_1 = \varepsilon\tau$ in definition of $\|(\cdot, \cdot)\|_\varepsilon$) and (χ_h, v_h) satisfies (5.74) then

$$\begin{aligned} & \|F'[\sigma^\varepsilon, u^\varepsilon](\chi_h - \tau I_{n \times n} v_h - D^2 v_h, 0)\|_{H^{-1}} \\ &\leq K_G h^\alpha \|(\chi_h - \tau I_{n \times n} v_h, v_h)\|_\varepsilon \\ &\leq K_G h^\alpha \left(\|(\chi_h, v_h)\|_\varepsilon + \sqrt{n}\tau(h\|v_h\|_{H^1} + \|v_h\|_{L^2}) \right) \\ &\leq CK_G h^\alpha \|(\chi_h, v_h)\|_\varepsilon. \end{aligned}$$

Hence, \tilde{F}' fulfills all [B2]–[B6]. The proof is complete. \square

CHAPTER 6

Applications

In the previous two chapters we have developed two abstract frameworks for conforming and mixed finite element approximations of the vanishing moment equation (2.9) under some (mild) structure conditions on the nonlinear differential operator F . The goal of this chapter is to apply the two abstract frameworks to three specific nonlinear PDEs, namely, the Monge-Ampère equation, the equation of the prescribed Gauss curvature, and the infinity-Laplacian equation. These three equations are chosen because they represent three different scenarios categorized by their linearizations, which are respectively, coercive, indefinite, and degenerate. It is shown that the abstract frameworks of Chapter 4 and 5 are broad enough to cover all three scenarios.

6.1. The Monge-Ampère equation

The Monge-Ampère equation (1.11) is without question the best known fully nonlinear second order PDE. It is to fully nonlinear second order PDEs as the Poisson equation is to linear second order PDEs. The Monge-Ampère equation arises from applications in differential geometry, optimal transportation, geophysics, antenna design, and astrophysics. We refer the reader to [19, 42, 44] and the references therein for more discussions about applications and PDE analysis of the Monge-Ampère equation.

In this section, we consider finite element approximations of the Monge-Ampère equation with Dirichlet boundary condition:

$$(6.1) \quad \det(D^2u) = f (> 0) \quad \text{in } \Omega,$$

$$(6.2) \quad u = g \quad \text{on } \partial\Omega.$$

A detailed analysis of conforming finite elements for the Monge-Ampère equation was carried out in [39] (also see [61]), where the authors proved optimal error estimates in the energy norm. The authors also studied mixed finite element methods for the Monge-Ampère equation in [38] (also see [61]) and obtained optimal error estimates for the scalar variable. However, we note that the results to be given below are sharper than those obtained in [38, 39] in the sense that weaker regularities of the solution u^ε are required in the error estimates and the dependence on ε^{-1} of the error bounds is less stringent.

In the case of the Monge-Ampère equation, we have

$$\begin{aligned} F(D^2u, \nabla u, u, x) &= f - \det(D^2u), \\ F'[v](w) &= -\text{cof}(D^2v) : D^2w, \\ F'[\mu, v](\kappa, w) &= -\text{cof}(\mu) : \kappa \end{aligned}$$

Remark 6.1. The inequality (4.5) implies that $F(D^2u, \nabla u, u, x) = f - \det(D^2u)$ instead of $\det(D^2u) - f$, which is used in most PDE literature [42]. Recall that we assume $-F$ is elliptic in the sense of [42, Chapter 17] in this book.

The vanishing moment approximation (2.9)–(2.11) becomes

$$(6.3) \quad -\varepsilon \Delta^2 u^\varepsilon + \det(D^2 u^\varepsilon) = f \quad \text{in } \Omega,$$

$$(6.4) \quad u^\varepsilon = g \quad \text{on } \partial\Omega,$$

$$(6.5) \quad \Delta u^\varepsilon = \varepsilon \quad \text{on } \partial\Omega,$$

and the linearization of

$$G_\varepsilon(u^\varepsilon) = \varepsilon \Delta^2 u^\varepsilon - \det(D^2 u^\varepsilon) + f$$

at the solution u^ε is

$$G'_\varepsilon[u^\varepsilon](v) = \varepsilon \Delta^2 v - \Phi^\varepsilon : D^2 v = \varepsilon \Delta^2 v - \operatorname{div}(\Phi^\varepsilon \nabla v),$$

where $\Phi^\varepsilon = \operatorname{cof}(D^2 u^\varepsilon)$, the cofactor matrix of the Hessian $D^2 u^\varepsilon$, and we have used Lemma 1.4 to obtain the last equality.

6.1.1. Conforming finite element methods for the Monge-Ampère equation. The finite element method for (6.3)–(6.5) is defined as follows (cf. (4.3)): find $u_h^\varepsilon \in V_g^h$ such that

$$(6.6) \quad -\varepsilon(\Delta u_h^\varepsilon, \Delta v_h) + (\det(D^2 u_h^\varepsilon), v_h) = (f, v_h) - \left\langle \varepsilon^2, \frac{\partial v_h}{\partial \nu} \right\rangle_{\partial\Omega} \quad \forall v_h \in V_0^h.$$

Recall $V = H^2(\Omega)$, and V_0^h and V_g^h are the C^1 finite element spaces of degree $k > 4$ defined by (4.2).

The goal of this section is to apply the abstract framework of Chapter 4 toward the finite element method (6.6) in two and three dimensions. Namely, we verify [A1]–[A5] and determine how the constants, C_i , δ , and $L(h)$, depend on ε . We summarize these results in the following theorem.

Theorem 6.2. *Let $u^\varepsilon \in H^s(\Omega)$ be the solution to (6.3)–(6.5) with $s \geq 3$ when $n = 2$ and $s > 3$ when $n = 3$. Then for $h \leq h_2$, there exists a unique solution $u_h^\varepsilon \in V_g^h$ to (6.6). Furthermore, there hold the following error estimates:*

$$(6.7) \quad \|u^\varepsilon - u_h^\varepsilon\|_{H^2} \leq C_7 h^{\ell-2} \|u^\varepsilon\|_{H^\ell},$$

$$(6.8) \quad \|u^\varepsilon - u_h^\varepsilon\|_{L^2} \leq C_8 \left(\varepsilon^{-\frac{1}{2}} h^\ell \|u^\varepsilon\|_{H^\ell} + C_7 L(h) h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right),$$

where

$$\begin{aligned} C_7 &= O\left(\varepsilon^{\frac{1}{2}(1-2n)}\right), & C_8 &= O\left(\varepsilon^{-\frac{1}{2}(5+2n)}\right), \\ L(h) &= O\left(\varepsilon^{\frac{5}{6}(2-n)} + h^{(2-n)}\right), & \ell &= \min\{s, k+1\}, \end{aligned}$$

and h_2 is chosen such that

$$h_2 \leq C \left(\varepsilon^{-\frac{1}{2}(1+2n)} \|u\|_{H^\ell} L(h_2) \right)^{\frac{1}{2-\ell}},$$

PROOF. We first state the a priori bounds shown in Chapter 3 (also see [36]):

$$(6.9) \quad \begin{aligned} \|u^\varepsilon\|_{H^j} &= O(\varepsilon^{\frac{1-j}{2}}) \quad (j = 1, 2, 3), \quad \|u^\varepsilon\|_{W^{j,\infty}} = O(\varepsilon^{1-j}) \quad (j = 1, 2), \\ \|\Phi^\varepsilon\|_{L^\infty} &= O(\varepsilon^{-1}), \quad \|\Phi^\varepsilon\|_{L^2} = O(\varepsilon^{-\frac{1}{2}}). \end{aligned}$$

We also note that by interpolation between L^p spaces, we have for $p \in [2, \infty]$

$$(6.10) \quad \|D^2 u^\varepsilon\|_{L^p} \leq \|D^2 u^\varepsilon\|_{L^2}^{\frac{2}{p}} \|D^2 u^\varepsilon\|_{L^\infty}^{\frac{p-2}{p}} = O(\varepsilon^{\frac{1-p}{p}}).$$

Next, since

$$\Delta^2 u^\varepsilon = \varepsilon^{-1} (\det(D^2 u^\varepsilon) - f),$$

by standard theory for the biharmonic equation, if $\partial\Omega$ is sufficiently smooth, then $u^\varepsilon \in H^4(\Omega)$ with

$$\begin{aligned} \|u\|_{H^4} &\leq \varepsilon^{-1} (\|\det(D^2 u^\varepsilon)\|_{L^2} + \|f\|_{L^2}) \\ &\leq \varepsilon^{-1} (\|D^2 u^\varepsilon\|_{L^{2n}}^n + \|f\|_{L^2}). \end{aligned}$$

Therefore, in view of (6.10), we have

$$(6.11) \quad \|u\|_{H^4} = O(\varepsilon^{-\frac{(1+2n)}{2}})$$

Thus, by (6.9), (6.11), and interpolation of Sobolev spaces, we have

$$(6.12) \quad \|u\|_{H^m} = O(\varepsilon^{\frac{n(2-m)-1}{2}}) \quad \forall m \in [2, 4].$$

In addition, u^ε is strictly convex. Hence, Φ^ε is positive definite, and therefore, there exists $C > 0$ such that

$$\langle \Phi^\varepsilon \nabla w, \nabla w \rangle \geq C \|\nabla w\|_{L^2}^2 \quad \forall w \in V_0.$$

It then follows that

$$(6.13) \quad \langle G'_\varepsilon[u^\varepsilon](w), w \rangle \geq C\varepsilon \|w\|_{H^2}^2 \quad \forall w \in V_0.$$

Next, using a Sobolev inequality

$$\begin{aligned} (6.14) \quad \|F'[u^\varepsilon]\|_{VV^*} &= \sup_{v \in V_0} \sup_{w \in V_0} \frac{\langle F'[u^\varepsilon](v), w \rangle}{\|v\|_{H^2} \|w\|_{H^2}} \\ &= \sup_{v \in V_0} \sup_{w \in V_0} \frac{(\Phi^\varepsilon \nabla v, \nabla w)}{\|v\|_{H^2} \|w\|_{H^2}} \\ &\leq \sup_{v \in V_0} \sup_{w \in V_0} \frac{\|\Phi^\varepsilon\|_{L^2} \|\nabla v\|_{L^4} \|\nabla w\|_{L^4}}{\|v\|_{H^2} \|w\|_{H^2}} \\ &\leq C \|\Phi^\varepsilon\|_{L^2} \leq C\varepsilon^{-\frac{1}{2}}. \end{aligned}$$

If $\partial\Omega$ is sufficiently smooth and $v \in V_0$ solves

$$\langle G'_\varepsilon[u^\varepsilon](v), w \rangle = (\varphi, w) \quad \forall w \in V_0,$$

where φ is some $L^2(\Omega)$ -function, then by standard elliptic PDE theory [32, 42], $v \in H^p(\Omega)$ for $p \geq 2$. Furthermore, in view of Remark 4.4 and the estimate

$$\left\| \frac{\partial F}{\partial r_{ij}}(u^\varepsilon) \right\|_{L^\infty} = \|\Phi_{ij}^\varepsilon\|_{L^\infty} \leq C\varepsilon^{-1},$$

we have

$$(6.15) \quad \|v\|_{H^3} \leq C\varepsilon^{-2} \|\varphi\|_{L^2}, \quad \|v\|_{H^4} \leq C\varepsilon^{-3} \|\varphi\|_{L^2}.$$

Thus, by (6.13)–(6.15), condition [A2] holds with

$$(6.16) \quad \begin{aligned} C_0 &\equiv 0, & C_1 &= O(\varepsilon), & C_2 &= O(\varepsilon^{-\frac{1}{2}}), \\ p &= 4, & C_R &= O(\varepsilon^{-3}), \end{aligned}$$

and therefore (cf. Theorems 4.2 and 4.3)

$$(6.17) \quad \begin{aligned} C_3 &= O(\varepsilon^{-1}), & C_4 &= O(\varepsilon^{-\frac{3}{2}}), \\ C_5 &= O(\varepsilon^{-5}), & h_0 &= 1. \end{aligned}$$

To confirm [A3]–[A4], we choose

$$Y = W^{2,2(n-1)}(\Omega), \quad \|\cdot\|_Y = \|\cdot\|_{W^{2,2(n-1)}}^{n-1}.$$

For a smooth function y , we use Lemma 1.4 and a Sobolev inequality to conclude

$$\begin{aligned} \frac{\|F'[y]\|_{VV^*}}{\|y\|_Y} &= \sup_{v \in V_0} \sup_{w \in V_0} \frac{\langle \text{cof}(D^2 y) : D^2 v, w \rangle}{\|y\|_Y \|v\|_{H^2} \|w\|_{H^2}} \\ &= \sup_{v \in V_0} \sup_{w \in V_0} \frac{(\text{cof}(D^2 y) \nabla v, \nabla w)}{\|y\|_Y \|v\|_{H^2} \|w\|_{H^2}} \\ &\leq C \left(\frac{\|\text{cof}(D^2 y)\|_{L^2}}{\|y\|_Y} \right) \leq C \left(\frac{\|D^2 y\|_{L^{2(n-1)}}^{n-1}}{\|y\|_Y} \right) \leq C. \end{aligned}$$

It then follows from a simple density argument that

$$\sup_{y \in Y} \frac{\|F'[y]\|_{VV^*}}{\|y\|_Y} \leq C,$$

and therefore condition [A3] holds, and by standard interpolation theory [22, 13] condition [A4] holds as well.

We also note that by (6.9)–(6.10) and Lemma 4.5

$$(6.18) \quad \|u^\varepsilon\|_Y \leq C\varepsilon^{\frac{1}{2}(3-2n)}, \quad C_6 = O\left(\varepsilon^{\frac{1}{2}(1-2n)}\right).$$

To verify [A5], we derive the following identity for any $v_h \in V_g^h$:

$$\begin{aligned} \|F'[u^\varepsilon] - F'[v_h]\|_{VV^*} &= \sup_{w \in V_0} \sup_{z \in V_0} \frac{\left((\text{cof}(D^2 u^\varepsilon) - \text{cof}(D^2 v_h)) \nabla w, \nabla z \right)}{\|w\|_{H^2} \|z\|_{H^2}} \\ &\leq C \|\text{cof}(D^2 u^\varepsilon) - \text{cof}(D^2 v_h)\|_{L^{\frac{3}{2}}} \end{aligned}$$

It follows that for $n = 2$,

$$\|F'[u^\varepsilon] - F'[v_h]\|_{VV^*} \leq C \|u^\varepsilon - v_h\|_{W^{2,\frac{3}{2}}} \leq C \|u^\varepsilon - v_h\|_{H^2}.$$

Hence, [A5] holds with $L(h) = C$.

For the case $n = 3$, we conclude by the mean value theorem that for any $i, j = 1, 2, 3$,

$$\begin{aligned} \|\text{cof}(D^2 u^\varepsilon)_{ij} - \text{cof}(D^2 v_h)_{ij}\|_{L^{\frac{3}{2}}} &= \|\det(D^2 u^\varepsilon|_{ij}) - \det(D^2 v_h|_{ij})\|_{L^{\frac{3}{2}}} \\ &\leq \|\Lambda^{ij}\|_{L^6} \|D^2 u^\varepsilon|_{ij} - D^2 v_h|_{ij}\|_{L^2} \\ &\leq \|\Lambda^{ij}\|_{L^6} \|u^\varepsilon - v_h\|_{H^2}, \end{aligned}$$

where $D^2 u^\varepsilon|_{ij}$ denotes the resulting 2×2 matrix after deleting the i^{th} row and j^{th} column of $D^2 u^\varepsilon$, and $\Lambda^{ij} = \text{cof}(D^2 u^\varepsilon|_{ij} + \gamma(D^2 v_h|_{ij} - D^2 u^\varepsilon|_{ij}))$ for some $\gamma \in [0, 1]$. Noting $\Lambda^{ij} \in \mathbf{R}^{2 \times 2}$, we have

$$\|\Lambda^{ij}\|_{L^6} \leq \|u^\varepsilon\|_{W^{2,6}} + \|v_h\|_{W^{2,6}}.$$

Thus, for any $\delta \in (0, 1)$ and $v_h \in V_g^h$ with $\|\mathcal{I}^h u^\varepsilon - v_h\|_{H^2} \leq \delta$, we have using the triangle inequality, the inverse inequality, and (6.10)

$$\begin{aligned} \|F'[u^\varepsilon] - F'[v_h]\|_{V^{*}} &\leq (\|u^\varepsilon\|_{W^{2,6}} + \|v_h\|_{W^{2,6}}) \|u^\varepsilon - v_h\|_{H^2} \\ &\leq C(\|u^\varepsilon\|_{W^{2,6}} + h^{-1} \|v_h - \mathcal{I}^h u^\varepsilon\|_{H^2}) \|u^\varepsilon - v_h\|_{H^2} \\ &\leq C(\varepsilon^{-\frac{5}{6}} + h^{-1} \delta) \|u^\varepsilon - v_h\|_{H^2} \\ &\leq C(\varepsilon^{-\frac{5}{6}} + h^{-1}) \|u^\varepsilon - v_h\|_{H^2} \\ &= L(h) \|u^\varepsilon - v_h\|_{H^2}. \end{aligned}$$

Thus, in the three-dimensional case [A5] holds with $L(h) = C(\varepsilon^{-\frac{5}{6}} + h^{-1})$.

Gathering all of our results, existence of a unique solution to (6.6) and the error estimates (6.7)–(6.8) follow from Theorem 4.7 and the estimates (6.16)–(6.18). \square

Remark 6.3. (a) Estimates (6.7) and (6.8) give the same asymptotic rates in h as those obtained in [39]. However, they provide an improvement to these previous results in the sense that the constants C_7 and h_2 have a better order dependence in terms of ε .

(b) We require stronger regularity in the three-dimensional case to ensure $L(h) = o(h^{2-\ell})$ (cf. Theorem 4.7).

6.1.2. Mixed finite element methods for the Monge-Ampère equation. The mixed finite element method for (6.3)–(6.5) is defined as follows (cf. (5.8)–(5.9)): find $(\sigma^\varepsilon, u^\varepsilon) \in W_\varepsilon^h \times Q_g^h$ such that

$$(6.19) \quad (\sigma_h^\varepsilon, \kappa_h) + b(\kappa_h, u_h^\varepsilon) = G(\kappa_h) \quad \forall \kappa_h \in W_0^h,$$

$$(6.20) \quad b(\sigma_h^\varepsilon, z_h) - \varepsilon^{-1} c(\sigma_h^\varepsilon, u_h^\varepsilon, z_h) = 0 \quad \forall z_h \in Q_0^h,$$

where

$$b(\kappa_h, u_h^\varepsilon) = (\text{div}(\kappa_h), \nabla u_h^\varepsilon), \quad c(\sigma_h^\varepsilon, u_h^\varepsilon, z_h) = (f - \det(\sigma_h^\varepsilon), z_h),$$

$G(\kappa_h)$ is defined by (5.7), $Q = H^1(\Omega)$, $W = [H^1(\Omega)]^{n \times n}$, and W_0^h , W_ε^h , Q_0^h , and Q_g^h are the Lagrange finite element spaces of degree $k \geq 2$ defined in Section 5.1.

We now apply the abstract theory developed in Chapter 5 to the mixed finite element method (6.19)–(6.20). Similar to the previous subsection, our goal is to show that assumptions [B1]–[B6] hold, and to explicitly derive how the constants, K_i , δ , and $R(h)$ depend on the parameter ε . We summarize our findings in the following theorem.

Theorem 6.4. *Suppose $u^\varepsilon \in H^s(\Omega)$ is the solution of (6.3)–(6.5) with $s \geq 3$ when $n = 2$ and $s > 4$ when $n = 3$. Furthermore, assume that $k \geq 4$ when $n = 3$. Then for $h \leq h_2$, there exists a unique solution $(\sigma_h^\varepsilon, u_h^\varepsilon) \in W_\varepsilon^h \times Q_g^h$ to (6.19)–(6.20). Furthermore, there hold the following error estimates:*

$$(6.21) \quad \|(\sigma^\varepsilon - \sigma_h^\varepsilon, u^\varepsilon - u_h^\varepsilon)\|_\varepsilon \leq K_8 h^{\ell-2} \|u^\varepsilon\|_{H^\ell}$$

$$(6.22) \quad \|u^\varepsilon - u_h^\varepsilon\|_{H^1} \leq K_{R_1} \left(K_9 h^{\ell-1} \|u^\varepsilon\|_{H^\ell} + K_8^2 R(h) h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right),$$

where

$$\begin{aligned} \|(\mu, v)\|_\varepsilon &= h\|\mu\|_{H^1} + \|\mu\|_{L^2} + \varepsilon^{-\frac{1}{2}}\|v\|_{H^1}, \\ K_8 &= O(\varepsilon^{\frac{1}{4}(22-15n)}), \quad K_9 = O(\varepsilon^{\frac{70-53n}{12}}), \\ R(h) &= O\left(|\log h|^{\frac{n-3}{2}} + (n-2)(\varepsilon^{-1}h^{-1} + h^{-2})\right), \\ \ell &= \min\{s, k+1\}, \end{aligned}$$

and h_2 is chosen such that

$$h_2 \approx \min\left\{\varepsilon^{\frac{1+4n}{6}}, \left(\varepsilon^{\frac{5}{4}(4-3n)}R(h_2)\|u^\varepsilon\|_{H^\ell}\right)^{\frac{1}{1-\ell}}, \left(\varepsilon^{-\frac{1}{2}}R(h_2)\|u^\varepsilon\|_{H^\ell}\right)^{\frac{1}{2-\ell}}\right\}.$$

PROOF. First, using the same arguments as those used to show condition [A2] in Theorem 6.2, we can also conclude that [B2] holds with

$$(6.23) \quad \begin{aligned} K_0 &\equiv 0, & K_1 &= O(1), & K_2 &= O(\varepsilon^{-1}), \\ p &= 4, & K_{R_0} &= O(\varepsilon^{-3}), & K_{R_1} &= O(\varepsilon^{-2}), \end{aligned}$$

and therefore (cf. Theorem 5.6 and Lemma 5.9)

$$(6.24) \quad K_4 = O(\varepsilon^{-\frac{3}{2}}), \quad K_5 = O(\varepsilon^{-\frac{7}{2}}), \quad K_7 = O(\varepsilon^{-\frac{1}{2}}).$$

To confirm [B3]–[B4], on noting that $F'[\mu, v](\kappa, w)$ is independent of v and w , we choose the spaces X and Y as follows:

$$\begin{aligned} X &= \left[L^{(n-1)(n+\varepsilon(3-n))}(\Omega)\right]^{n \times n}, \quad Y = \emptyset, \\ \|(\omega, y)\|_{X \times Y} &= \|\omega\|_{L^{(n-1)(n+\varepsilon(3-n))}}^{n-1} \quad \forall \omega \in X, y \in Y. \end{aligned}$$

Then using a Sobolev inequality, we have for all $\omega \in X$, $y \in Y$, $\chi \in W$, $v \in Q$

$$\begin{aligned} \|F'[\omega, y](\chi, v)\|_{H^{-1}} &= \sup_{w \in Q_0} \frac{(\text{cof}(\omega) : \chi, w)}{\|w\|_{H^1}} \\ &\leq C \|\text{cof}(\omega)\|_{L^{n+\varepsilon(3-n)}} \|\chi\|_{L^2} \\ &\leq C \|\omega\|_{L^{(n-1)(n+\varepsilon(3-n))}}^{n-1} \|\chi\|_{L^2} \\ &\leq C \|(\omega, y)\|_{X \times Y} (\|\chi\|_{L^2} + \|v\|_{H^1}). \end{aligned}$$

Thus condition [B3] holds.

To confirm [B4], we note that by the inverse inequality, standard stability results for the interpolation operator, and (5.12) to conclude that if $\sigma^\varepsilon \in [H^{s-2}(\Omega)]^{n \times n}$ then for any $p \in [2, \infty]$ and $\ell \in [3, \min\{s, k+1\}]$

$$\begin{aligned} (6.25) \quad \|\Pi^h \sigma^\varepsilon\|_{L^p} &\leq \|\Pi^h \sigma^\varepsilon - \mathcal{I}^h \sigma^\varepsilon\|_{L^p} + \|\mathcal{I}^h \sigma^\varepsilon\|_{L^p} \\ &\leq C \left(h^{\frac{n}{p} - \frac{n}{2}} \|\Pi^h \sigma^\varepsilon - \mathcal{I}^h \sigma^\varepsilon\|_{L^2} + \|\mathcal{I}^h \sigma^\varepsilon\|_{L^p} \right) \\ &\leq C \left(h^{\frac{n}{p} - \frac{n}{2} + \ell - 2} \|\sigma^\varepsilon\|_{H^{\ell-2}} + \|\sigma^\varepsilon\|_{L^p} \right). \end{aligned}$$

Therefore, for any $\gamma \in [0, 1]$

$$\begin{aligned} &\|(\Pi^h \sigma^\varepsilon - \gamma \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon - \gamma u^\varepsilon)\|_{X \times Y} \\ &= \|\Pi^h \sigma^\varepsilon - \gamma \sigma^\varepsilon\|_{L^{6(n-1)}}^{n-1} \\ &\leq C \left(h^{\frac{n}{6(n-1)} - \frac{n}{2} + \ell - 2} \|\sigma^\varepsilon\|_{H^{\ell-2}} + \|\sigma^\varepsilon\|_{L^{6(n-1)}} \right)^{n-1}. \end{aligned}$$

For the two-dimensional case, we set $\ell = 3$ and use (6.9)–(6.10) to get

$$\begin{aligned} \|(\Pi^h \sigma^\varepsilon - \gamma \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon - \gamma u^\varepsilon)\|_{X \times Y} &\leq C(h^{\frac{1}{3}} \|\sigma^\varepsilon\|_{H^1} + \|\sigma^\varepsilon\|_{L^6}) \\ &\leq C(h^{\frac{1}{3}} \varepsilon^{-1} + \varepsilon^{-\frac{5}{6}}) = O(\varepsilon^{-1}). \end{aligned}$$

For the three-dimensional case, we set $\ell = \frac{13}{4}$ and use (6.10)–(6.12) to get

$$\begin{aligned} \|(\Pi^h \sigma^\varepsilon - \gamma \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon - \gamma u^\varepsilon)\|_{X \times Y} &\leq C(\|\sigma^\varepsilon\|_{H^{\frac{5}{4}}} + \|\sigma^\varepsilon\|_{L^{12}})^2 \\ &\leq C(\varepsilon^{-\frac{19}{4}} + \varepsilon^{-\frac{11}{6}}) = O(\varepsilon^{-\frac{19}{4}}). \end{aligned}$$

Therefore by Lemma 5.8

$$(6.26) \quad K_3 = O\left(\varepsilon^{\frac{1}{4}(26-15n)}\right), \quad K_6 = O\left(\varepsilon^{\frac{1}{4}(22-15n)}\right).$$

To confirm [B5], we have for any $(\mu_h, v_h) \in W_\varepsilon^h \times Q_g^h$, $(\kappa_h, z_h) \in W^h \times Q^h$, and $w_h \in Q^h$

$$\begin{aligned} \left\langle (F'[\sigma^\varepsilon, u^\varepsilon] - F'[\mu_h, v_h])(\kappa_h, z_h), w_h \right\rangle &= \left((\text{cof}(\sigma^\varepsilon) - \text{cof}(\mu_h)) : \kappa_h, w_h \right) \\ &\leq C |\log h|^{\frac{3-n}{2}} h^{1-\frac{n}{2}} \|\text{cof}(\sigma^\varepsilon) - \text{cof}(\mu_h)\|_{L^2} \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1}, \end{aligned}$$

where we have used the inverse inequality [13, Lemma 4.9.1].

If $n = 2$, then $\|\text{cof}(\sigma^\varepsilon) - \text{cof}(\mu_h)\|_{L^2} = \|\sigma^\varepsilon - \mu_h\|_{L^2}$, and so condition [B5] holds with $R(h) = C |\log h|^{\frac{1}{2}}$. For $n = 3$,

$$\begin{aligned} \|(\text{cof}(\sigma^\varepsilon) - \text{cof}(\mu_h))_{ij}\|_{L^2} &= \|\det(\sigma^\varepsilon|_{ij}) - \det(\mu_h|_{ij})\|_{L^2} \\ &= \|\Lambda^{ij} : (\sigma^\varepsilon|_{ij} - \mu_h|_{ij})\|_{L^2} \\ &\leq \|\Lambda^{ij}\|_{L^\infty} \|\sigma^\varepsilon|_{ij} - \mu_h|_{ij}\|_{L^2} \\ &\leq C \|\Lambda^{ij}\|_{L^\infty} \|\sigma^\varepsilon - \mu_h\|_{L^2}, \end{aligned}$$

where $\Lambda^{ij} = \text{cof}(\sigma^\varepsilon|_{ij} + \gamma(\mu_h|_{ij} - \sigma^\varepsilon|_{ij}))$ for some $\gamma \in [0, 1]$, and we have used the same notation as in Section 6.1.1. Since $\Lambda^{ij} \in \mathbf{R}^{2 \times 2}$, we have for $\|\Pi^h \sigma^\varepsilon - \mu_h\|_{L^2} \leq \delta \in (0, 1)$

$$\begin{aligned} \|\Lambda^{ij}\|_{L^\infty} &\leq C(\|\sigma^\varepsilon + \Pi^h \sigma^\varepsilon\|_{L^\infty} + \|\Pi^h \sigma^\varepsilon - \mu_h\|_{L^\infty}) \\ &\leq C(\varepsilon^{-1} + h^{-\frac{3}{2}} \delta) \leq C(\varepsilon^{-1} + h^{-\frac{3}{2}}). \end{aligned}$$

It then follows that [B5] holds in the case $n = 3$ with $R(h) = C(\varepsilon^{-1} h^{-\frac{1}{2}} + h^{-2})$. We note that for the hypotheses in Theorems 5.10–5.11 to hold, we require $R(h) = o(h^{2-\ell})$ as $h \rightarrow 0^+$ for fixed ε . This requirement is satisfied if $\ell > 2$ in two dimension, and this bound is true provided $\ell > 4$ in three dimensions.

Next, to verify condition [B6], we first use Holder's inequality and (6.9)–(6.10) to conclude that for $p \in [1, 2]$ and any $i, j, k = 1, 2, \dots, n$ ($n = 2, 3$)

$$\left\| \frac{\partial \Phi_{ij}^\varepsilon}{\partial x_k} \right\|_{L^p} \leq C \|D^2 u^\varepsilon\|_{L^{\frac{2p}{2-p}}}^{n-2} \|D^2 u^\varepsilon\|_{H^1} = O\left(\varepsilon^{\frac{(2-3p)(n-2)-2p}{2p}}\right).$$

Therefore, in view of Proposition 5.4 and the estimates

$$\begin{aligned} \left\| \frac{\partial F}{\partial r_{ij}} \right\|_{L^\infty} &= \|\Phi_{ij}^\varepsilon\|_{L^\infty} = O(\varepsilon^{-1}), \\ \left\| \frac{\partial F}{\partial r_{ij}} \right\|_{W^{1, \frac{6}{5}}} &= \|\Phi_{ij}^\varepsilon\|_{W^{1, \frac{6}{5}}} = O\left(\varepsilon^{\frac{1-2n}{3}}\right), \end{aligned}$$

to conclude that [B6] holds with

$$(6.27) \quad \alpha = 1, \quad K_G = \varepsilon^{\frac{1-2n}{3}}.$$

Thus, the existence of a unique solution $(\sigma_h^\varepsilon, u_h^\varepsilon)$ to (6.19)–(6.20) and the error estimates (6.21)–(6.22) follows from Theorem 5.10 and the estimates (6.23)–(6.27). \square

Remark 6.5. The error estimates in Theorem 6.4 have the same order of convergence in h as the estimates derived in [38], but the constants' dependence on ε in (6.21)–(6.22) are sharper than these previous results.

6.1.3. Numerical experiments and rates of convergence. Extensive numerical experiments for the finite element methods (6.6) and (6.19)–(6.20) in the two-dimensional setting have already been reported in [39] and [38], respectively. These tests confirmed the error estimates (6.7)–(6.8) and (6.21)–(6.22), and indicate that these estimates are sharp. Furthermore, the tests confirm the following rates of convergence:

$$\|u - u^\varepsilon\|_{L^2} = O(\varepsilon), \quad \|u - u^\varepsilon\|_{H^1} = O(\varepsilon^{\frac{3}{4}}), \quad \|u - u^\varepsilon\|_{H^2} = O(\varepsilon^{\frac{1}{4}}),$$

which are proved in Theorem 3.19 when the viscosity solution u belongs to the space $W^{2,\infty}(\Omega) \cap H^3(\Omega)$.

In this section, we expand on these earlier results, performing two and three-dimensional numerical experiments and comparing the results with these earlier findings. We also show that for certain problems, one must choose an appropriate $h - \varepsilon$ relation in order for the method to converge. The tests below are done on the unit square $\Omega = (0, 1)^n$ ($n = 1, 2, 3$).

Test 6.1.1. For this test, we calculate $\|u - u_h^\varepsilon\|$ for fixed $h = 0.02$, while varying ε in order to estimate $\|u - u^\varepsilon\|$. We solve the mixed finite element method (6.19)–(6.20) using the quadratic Lagrange finite element ($k = 2$) with the following test functions:

$$\begin{aligned} \text{(a)} \quad u &= e^{(x_1^2 + x_2^2 + x_3^2)/2}, \quad f = (1 + x_1^2 + x_2^2 + x_3^2)e^{3(x_1^2 + x_2^2 + x_3^2)/2}, \\ \text{(b)} \quad u &= x_1^2 + x_2^2 + x_3^2, \quad f = 8. \end{aligned}$$

After having computed the solution, we list the errors in various norms in Table 1 and plot the results in Figures 2–3. The figures indicate that

$$\begin{aligned} \|u - u_h^\varepsilon\|_{L^\infty} &= O(\varepsilon), & \|u - u_h^\varepsilon\|_{L^2} &= O(\varepsilon), \\ \|u - u_h^\varepsilon\|_{H^1} &= O(\varepsilon^{\frac{3}{4}}), & \|\sigma^\varepsilon - \sigma_h^\varepsilon\|_{L^2} &= O(\varepsilon^{\frac{1}{4}}). \end{aligned}$$

Therefore, since h is small, we expect

$$\begin{aligned} \|u - u^\varepsilon\|_{L^\infty} &\approx O(\varepsilon), & \|u - u^\varepsilon\|_{L^2} &\approx O(\varepsilon), \\ \|u - u^\varepsilon\|_{H^1} &\approx O(\varepsilon^{\frac{3}{4}}), & \|u - u^\varepsilon\|_{H^2} &\approx O(\varepsilon^{\frac{1}{4}}). \end{aligned}$$

We note that these are the same rates of convergence found in [39] and [38].

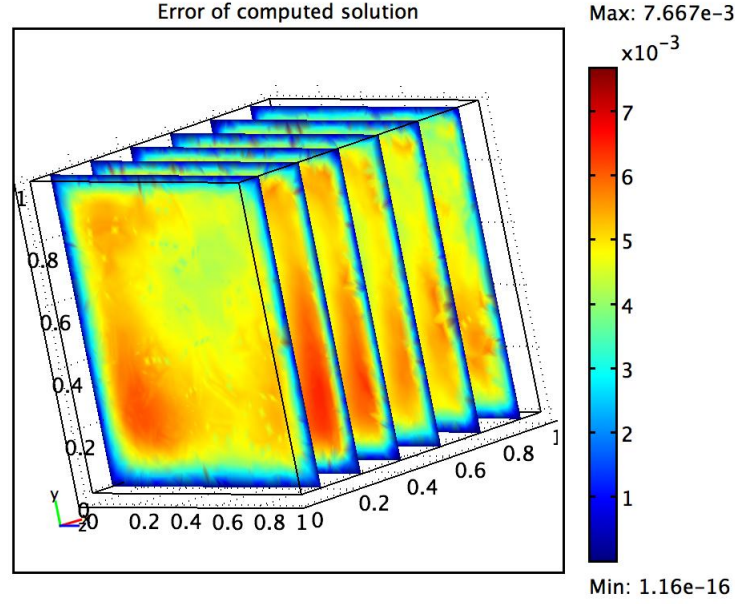


FIGURE 1. Test 6.1.1a. Error of computed solution with $\varepsilon = 0.01$ and $h = 0.02$.

TABLE 1. Test 6.1.1. Error of $\|u - u_h^\varepsilon\|$ w.r.t. ε ($h = 0.02$)

	ε	$\ u - u_h^\varepsilon\ _{L^\infty}(\text{rate})$	$\ u - u_h^\varepsilon\ _{L^2}(\text{rate})$	$\ u - u_h^\varepsilon\ _{H^1}(\text{rate})$	$\ \sigma - \sigma_h^\varepsilon\ _{L^2}(\text{rate})$
Test 6.1.1a	5.0E-01	1.19E-01(—)	5.71E-02(—)	3.47E-01(—)	3.34E+00(—)
	2.5E-01	8.91E-02(0.42)	4.63E-02(0.30)	2.88E-01(0.27)	3.08E+00(0.12)
	1.0E-01	5.36E-02(0.55)	3.19E-02(0.41)	2.09E-01(0.35)	2.72E+00(0.14)
	5.0E-02	2.35E-02(1.19)	1.59E-02(1.00)	1.21E-01(0.79)	2.29E+00(0.25)
	2.5E-02	1.18E-02(0.99)	8.95E-03(0.83)	7.35E-02(0.72)	1.99E+00(0.20)
	1.0E-02	5.57E-03(0.82)	4.25E-03(0.81)	3.91E-02(0.69)	1.66E+00(0.20)
Test 6.1.1b	5.0E-01	1.61E-01(—)	7.47E-02(—)	4.27E-01(—)	3.12E+00(—)
	2.5E-01	1.36E-01(0.24)	6.48E-02(0.21)	3.75E-01(0.19)	2.91E+00(0.10)
	1.0E-01	7.94E-02(0.59)	4.17E-02(0.48)	2.52E-01(0.43)	2.36E+00(0.23)
	5.0E-02	4.20E-02(0.92)	2.49E-02(0.74)	1.61E-01(0.64)	1.92E+00(0.29)
	2.5E-02	1.99E-02(1.08)	1.36E-02(0.88)	9.70E-02(0.73)	1.57E+00(0.29)
	1.0E-02	7.36E-03(1.09)	5.76E-03(0.94)	4.85E-02(0.76)	1.26E+00(0.24)
	5.0E-03	3.79E-03(0.96)	3.10E-03(0.89)	2.97E-02(0.71)	1.11E+00(0.17)

Test 6.1.2. The purpose of this test is to calculate the rate of convergence of $\|u^\varepsilon - u_h^\varepsilon\|$ for fixed $\varepsilon=0.001$ in various norms. We solve problem (6.19)–(6.20) using the linear Lagrange element ($k = 1$), but with the boundary condition $\sigma_h^\varepsilon \nu \cdot \nu|_{\partial\Omega} = \varepsilon$

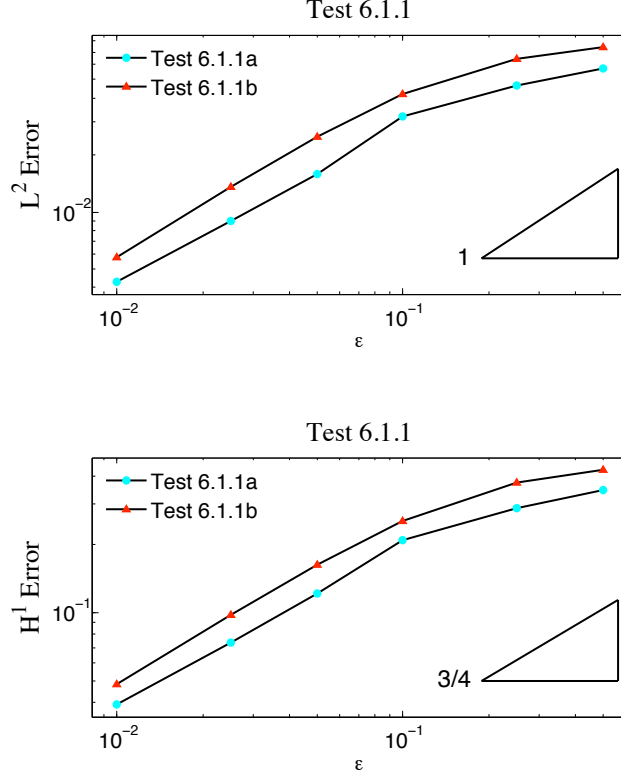


FIGURE 2. Test 6.1.1. Error $\|u - u_h^\varepsilon\|_{L^\infty}$ (top) and $\|u - u_h^\varepsilon\|_{L^2}$ (bottom) w.r.t. ε ($h = 0.02$).

replaced by $\sigma_h^\varepsilon \nu \cdot \nu|_{\partial\Omega} = \phi^\varepsilon$. We use the following test functions and data:

$$\begin{aligned}
 \text{(a)} \quad & u^\varepsilon = x_1^2 + x_2^2 + x_3^2, & f^\varepsilon &= 8, \\
 & g^\varepsilon = x_1^2 + x_2^2 + x_3^2, & \phi^\varepsilon &= 2, \\
 \text{(b)} \quad & u^\varepsilon = x_1^4 + x_2^2 + x_3^6, & f^\varepsilon &= 720x_1^2x_3^4 - \varepsilon 8640x_3^2, \\
 & g^\varepsilon = x_1^4 + x_2^2 + x_3^6, & \phi^\varepsilon &= 12x_1^2\nu_1^2 + 2\nu_2^2 + 30x_3^4\nu_3^2.
 \end{aligned}$$

After computing the solution, we list the errors in Table 2 and plot the results in Figure 4. We note that the mixed finite element theory in the preceding sections was only developed for $k \geq 2$. However, our numerical experiments also indicate that the method works for the case $k = 1$. Indeed, the tests indicate the following rates of convergence:

$$\|u^\varepsilon - u_h^\varepsilon\|_{L^2} = O(h^2), \quad \|u^\varepsilon - u_h^\varepsilon\|_{H^1} = O(h).$$

Test 6.1.3. The purpose of this test is to calculate the error $\|u^\varepsilon - u_h^\varepsilon\|$ in various norms using a fixed $h - \varepsilon$ relation. We solve the finite element method (6.6) in two dimensions with V^h denoting the Argyris finite element space of degree five [22], and replace the boundary condition $\Delta u_h^\varepsilon|_{\partial\Omega} = \varepsilon$ by $\Delta u_h^\varepsilon|_{\partial\Omega} = \phi^\varepsilon$. We use the

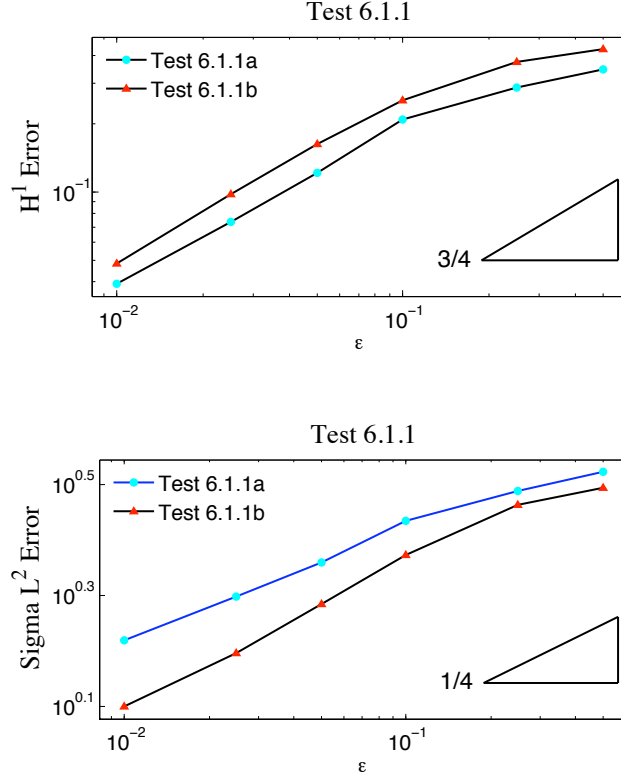


FIGURE 3. Test 6.1.1. Error $\|u - u_h^\varepsilon\|_{H^1}$ (top) and $\|\sigma - \sigma_h^\varepsilon\|_{L^2}$ (bottom) w.r.t. ε ($h = 0.02$).

TABLE 2. Test 6.1.2. Error of $\|u^\varepsilon - u_h^\varepsilon\|$ w.r.t. h ($\varepsilon=0.001$, linear Lagrange element)

	h	$\ u^\varepsilon - u_h^\varepsilon\ _{L^2}$	$\ u^\varepsilon - u_h^\varepsilon\ _{H^1}$	$\ \sigma^\varepsilon - \sigma_h^\varepsilon\ _{L^2}$
Test 6.1.2a	1.75E-01	4.65E-02(—)	2.46E-01(—)	7.57E-01(—)
	1.25E-01	2.25E-02(2.16)	1.72E-01(1.07)	8.75E-01(-0.43)
	7.50E-02	7.95E-03(2.03)	1.04E-01(0.99)	8.39E-01(0.08)
	6.00E-02	5.13E-03(1.97)	8.07E-02(1.12)	6.61E-01(1.07)
	4.00E-02	1.97E-03(2.35)	5.28E-02(1.05)	5.85E-01(0.30)
	2.00E-02	1.13E-03(0.80)	4.17E-02(0.34)	5.28E-01(0.15)
Test 6.1.2b	1.75E-01	1.04E-01(—)	8.72E-01(—)	3.91E+00(—)
	1.25E-01	5.46E-02(1.92)	6.80E-01(0.74)	3.92E+00(-0.01)
	7.50E-02	1.97E-02(1.99)	4.26E-01(0.92)	3.75E+00(0.09)
	6.00E-02	1.30E-02(1.85)	3.40E-01(1.01)	3.33E+00(0.53)
	4.00E-02	7.57E-03(1.34)	2.29E-01(0.97)	3.25E+00(0.06)
	2.00E-02	8.43E-03(-0.16)	1.85E-01(0.31)	3.04E+00(0.09)

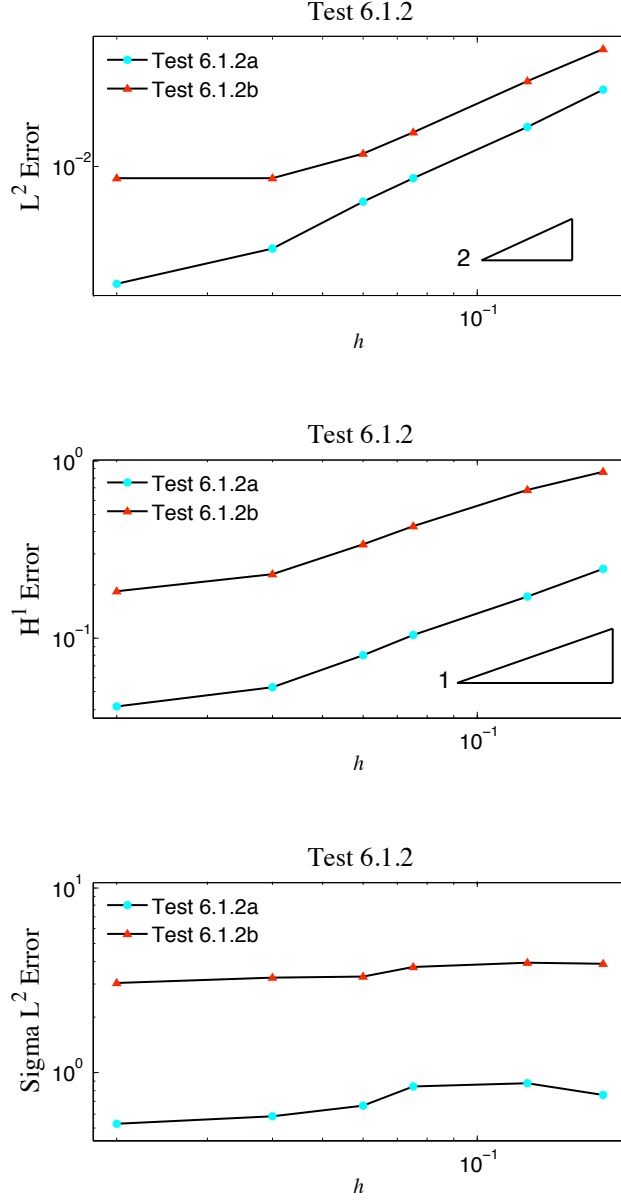


FIGURE 4. Test 6.1.2. Error $\|u^\varepsilon - u_h^\varepsilon\|_{L^2}$ (top), $\|u^\varepsilon - u_h^\varepsilon\|_{H^1}$ (middle), and $\|\sigma^\varepsilon - \sigma_h^\varepsilon\|_{L^2}$ (bottom) w.r.t. h ($\varepsilon = 0.001$).

following test function and data:

$$u^\varepsilon = -\sqrt{r^\varepsilon - (x_1^2 + x_2^2)}, \quad f^\varepsilon = \frac{r^\varepsilon}{(r^\varepsilon - (x_1^2 + x_2^2))^2} - \varepsilon \frac{(x_1^2 + x_2^2)(8r^\varepsilon - x_1^2 - x_2^2) + 8(r^\varepsilon)^2}{(r - (x_1^2 + x_2^2))^{\frac{7}{2}}},$$

$$g^\varepsilon = -\sqrt{r^\varepsilon - (x_1^2 + x_2^2)}, \quad \phi^\varepsilon = \frac{2r^\varepsilon - (x_1^2 + x_2^2)}{(r^\varepsilon - (x_1^2 + x_2^2))^{\frac{3}{2}}},$$

$$r^\varepsilon = 2 + \varepsilon.$$

On the domain $\Omega = (0, 1)^2$, $u^\varepsilon \in C^\infty(\Omega)$ for any $\varepsilon > 0$. However, the limiting function

$$u := \lim_{\varepsilon \rightarrow 0^+} u^\varepsilon = -\sqrt{2 - (x_1^2 + x_2^2)}$$

is not smooth, and in fact, there only holds $u \in W^{1,p}$ where $p \in [1, 4)$ (cf. [29]).

We solve (6.6) using the following four $h - \varepsilon$ relations:

$$\begin{aligned} h &= 2\varepsilon^{\frac{3}{2}}, & h &= \varepsilon, \\ h &= 0.5\varepsilon^{\frac{1}{2}}, & h &= 0.5\varepsilon^{\frac{1}{4}}. \end{aligned}$$

We list the errors of the computed solution in Table 3 and plot the results in Figures 6–7.

Since $\|u^\varepsilon\|_{H^\ell} \rightarrow \infty$ for any $\ell \geq 2$ as $\varepsilon \rightarrow 0^+$, we suspect that a stringent $h - \varepsilon$ relation will be needed in order for the method to converge in view of the error estimates (6.7)–(6.8). This supposition is verified by the numerical tests, as the method does not converge in any norm using the relation $h = 0.5\varepsilon^{\frac{1}{4}}$. Furthermore, we observe that the method does not converge in the H^2 -norm for any $h - \varepsilon$ relations used in the experiments. This behavior is expected since the limiting solution u is not in this space. We plot the error of the computed solution in Figure 5 with parameters $\varepsilon = h = 0.04$. As seen from the picture, the error is concentrated at the singularity of u .

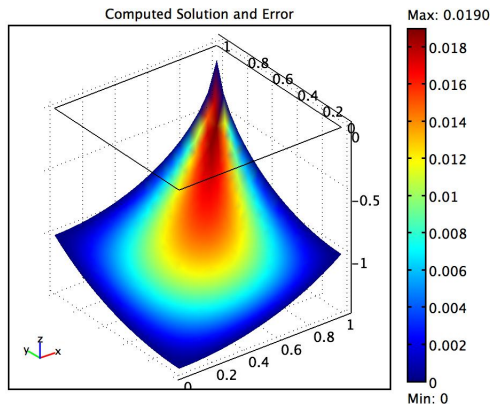


FIGURE 5. Test 6.1.3. Error of computed solution with $\varepsilon = 0.04$, $h = 0.04$.

Test 6.1.4. For our last test, we numerically back up the theoretical results given in Chapter 3, that is, we compute the vanishing moment approximation (2.9)–(2.11)₁ in the radial symmetric case. To this end, we solve (3.6)–(3.9) in the domain $\Omega = (0, 1)$. We use the Hermite cubic finite element to construct our finite element space, and we use the following data:

$$f = (1 + r^2)e^{nr^2/2}, \quad g(1) = e^{\frac{1}{2}}.$$

It can be readily checked that the exact solution is $u = e^{r^2/2}$.

We plot the computed solution and corresponding error in Figure 8 with parameters $n = 4$, $\varepsilon = 10^{-1}$, $h = 4.0 \times 10^{-3}$. We also plot the computed Laplacian,

TABLE 3. Test 6.1.3. Error of $\|u^\varepsilon - u_h^\varepsilon\|$ with $h - \varepsilon$ relation

	ε	h	$\ u^\varepsilon - u_h^\varepsilon\ _{L^\infty}$	$\ u^\varepsilon - u_h^\varepsilon\ _{L^2}$	$\ u^\varepsilon - u_h^\varepsilon\ _{H^1}$	$\ u^\varepsilon - u_h^\varepsilon\ _{L^2}$
$h = 2\varepsilon^{\frac{3}{2}}$	2.00E-01	1.79E-01	3.94E-02	2.02E-02	9.41E-02	6.32E-01
	1.00E-01	6.32E-02	4.11E-02	2.10E-02	1.01E-01	7.55E-01
	5.00E-02	2.24E-02	3.45E-02	1.76E-02	8.85E-02	7.84E-01
	4.00E-02	1.60E-02	3.12E-02	1.59E-02	8.17E-02	8.17E-01
$h = \varepsilon$	2.00E-01	2.00E-01	3.96E-02	2.03E-02	9.54E-02	8.79E-01
	1.00E-01	1.00E-01	4.12E-02	2.11E-02	1.02E-01	1.11E+00
	5.00E-02	5.00E-02	3.45E-02	1.76E-02	8.89E-02	1.34E+00
	4.00E-02	4.00E-02	3.13E-02	1.59E-02	8.23E-02	1.73E+00
	2.50E-02	2.50E-02	2.38E-02	1.21E-02	6.58E-02	1.88E+00
	1.25E-02	1.25E-02	1.40E-02	7.11E-03	4.26E-02	2.41E+00
$h = 0.5\varepsilon^{\frac{1}{2}}$	2.00E-01	2.24E-01	3.95E-02	2.02E-02	9.45E-02	6.70E-01
	1.00E-01	1.58E-01	4.14E-02	2.12E-02	1.03E-01	1.18E+00
	5.00E-02	1.12E-01	3.63E-02	1.84E-02	9.92E-02	2.74E+00
	4.00E-02	1.00E-01	3.33E-02	1.69E-02	9.53E-02	3.31E+00
	2.50E-02	7.91E-02	2.67E-02	1.33E-02	8.64E-02	4.63E+00
	1.25E-02	5.59E-02	1.90E-02	8.14E-03	7.25E-02	6.39E+00
	6.25E-03	3.95E-02	1.96E-02	4.47E-03	6.91E-02	1.10E+01
$h = 0.5\varepsilon^{\frac{1}{4}}$	2.00E-01	3.34E-01	4.04E-02	2.08E-02	1.02E-01	1.18E+00
	1.00E-01	2.81E-01	4.32E-02	2.21E-02	1.14E-01	1.62E+00
	5.00E-02	2.36E-01	4.17E-02	2.09E-02	1.26E-01	2.79E+00
	4.00E-02	2.24E-01	4.40E-02	2.14E-02	1.42E-01	3.50E+00
	2.50E-02	1.99E-01	5.89E-02	2.49E-02	1.96E-01	6.03E+00
	1.25E-02	1.67E-01	6.15E-02	2.10E-02	2.03E-01	7.33E+00

$\Delta u^\varepsilon := u_{rr}^\varepsilon + \frac{2}{r}u_r^\varepsilon$, as well. As shown by the pictures, the vanishing moment methodology accurately captures the convex solution in higher dimensions. Also, as expected, the Laplacian of u^ε is strictly positive (cf. Theorem 3.9).

Next, we plot both u_r and u_{rr} in two and four dimensions in Figures 9–10 with ε -values, 10^{-1} , 10^{-3} , 10^{-5} . Recall that the Hessian matrix of u^ε only has two distinct eigenvalues u_{rr}^ε and $\frac{1}{r}u_r^\varepsilon$. As seen in Figure 9, u_r^ε is positive for all ε -values and for both dimensions $n = 2$ and $n = 4$. This result is in accordance with Corollary 3.7. Finally, Figure 10 shows that u_{rr}^ε is strictly positive except for a small ε -neighborhood of the boundary, which agrees with the theoretical results established in Theorem 3.12.

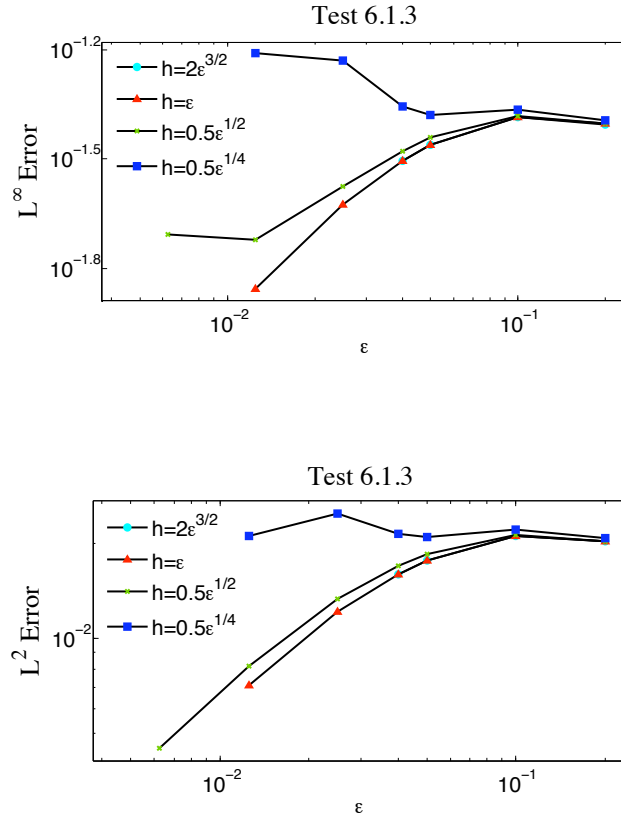


FIGURE 6. Test 6.1.3. Error $\|u^\epsilon - u_h^\epsilon\|_{L^\infty}$ (top) and $\|u^\epsilon - u_h^\epsilon\|_{L^2}$ (bottom) with various $h - \epsilon$ relations.

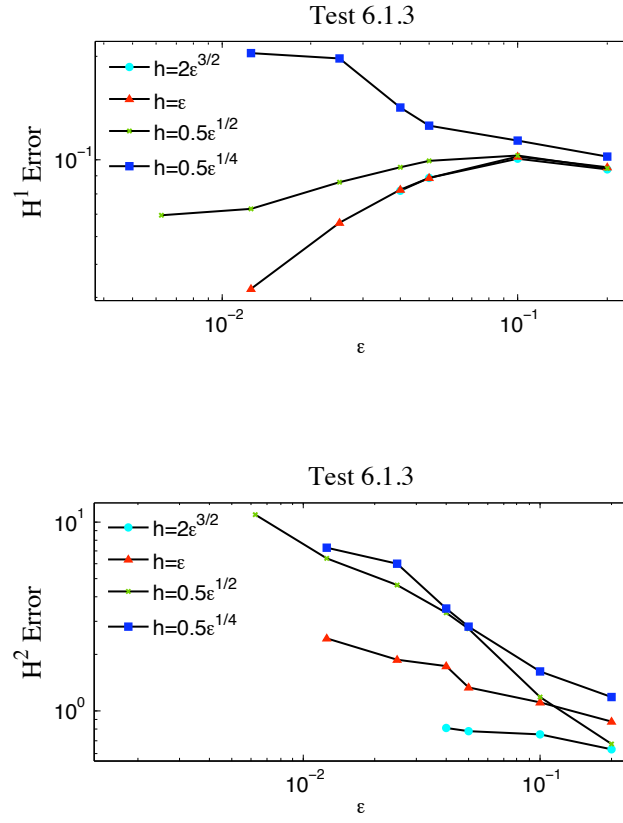


FIGURE 7. Test 6.1.3. Error $\|u^\epsilon - u_h^\epsilon\|_{H^2}$ (top) and $\|u^\epsilon - u_h^\epsilon\|_{H^2}$ (bottom) with various $h - \epsilon$ relations.

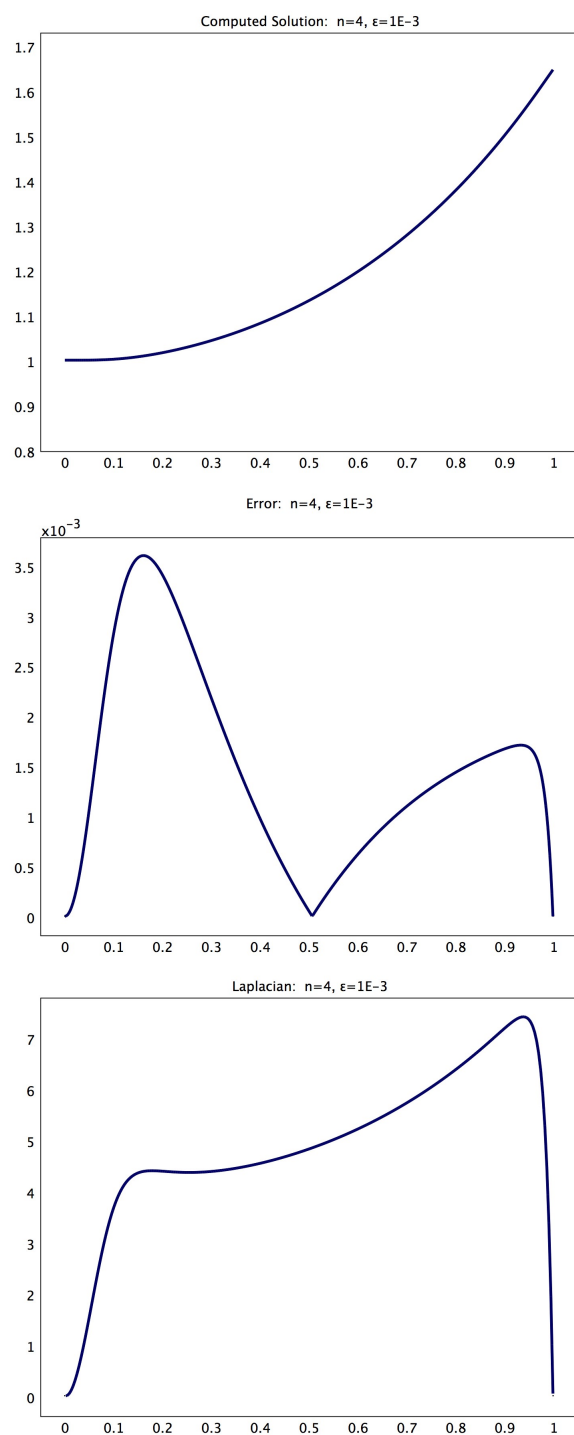


FIGURE 8. Test 6.1.4. Computed solution of (3.6)–(3.8) (top), error (middle), and computed Laplacian (bottom) with $n = 4$, $\varepsilon = 10^{-1}$, $h = 4 \times 10^{-3}$.

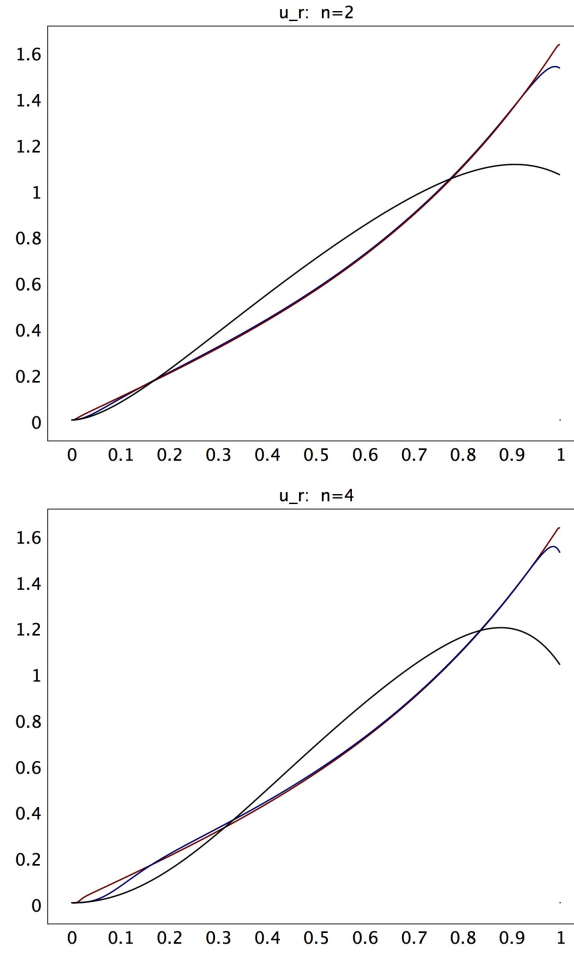


FIGURE 9. Test 6.1.4. Computed u_r of (3.6)–(3.8) for $n = 2$ (top), and $n = 4$ (bottom) with $\varepsilon = 10^{-1}$ (black), $\varepsilon = 10^{-3}$ (blue), and $\varepsilon = 10^{-5}$ (red) ($h = 4 \times 10^{-3}$).

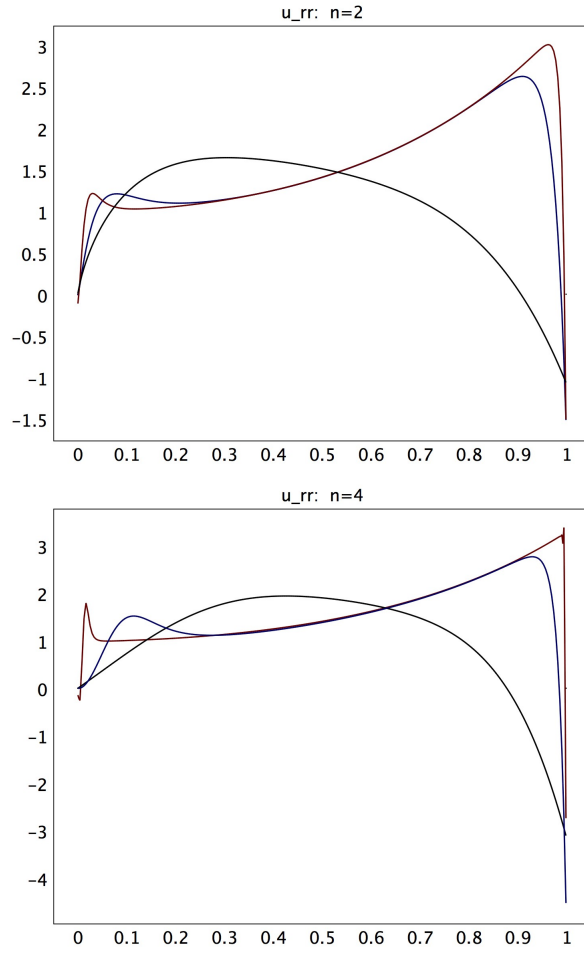


FIGURE 10. Test 6.1.4. Computed u_{rr} of (3.6)–(3.8) for $n = 2$ (top), and $n = 4$ (bottom) with $\varepsilon = 10^{-1}$ (black), $\varepsilon = 10^{-3}$ (blue), and $\varepsilon = 10^{-5}$ (red) ($h = 4 \times 10^{-3}$).

6.2. The equation of prescribed Gauss curvature

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain and $g \in C^0(\partial\Omega)$. For a given constant $\mathcal{K} > 0$, the simplest version of the famous Minkowski problem (cf. [47, 42]) asks to find a function u whose graph (as a manifold) has the constant Gauss curvature \mathcal{K} and u takes the boundary value g on $\partial\Omega$. The Gauss curvature of the graph of u is given by the formula

$$\frac{\det(D^2u)}{(1 + |\nabla u|^2)^{\frac{n+2}{2}}},$$

and therefore, if such a function exists, it must satisfy

$$(6.28) \quad \det(D^2u) = \mathcal{K}(1 + |\nabla u|^2)^{\frac{n+2}{2}} \quad \text{in } \Omega,$$

$$(6.29) \quad u = g \quad \text{on } \partial\Omega.$$

The equation (6.28), which is called *the equation of prescribed Gauss curvature*, is a fully nonlinear Monge-Ampère-type equation.

It is known [47] that there exists a constant $\mathcal{K}^* > 0$ such that for each $\mathcal{K} \in [0, \mathcal{K}^*)$, problem (6.28)–(6.29) has a unique convex viscosity solution. Theoretically, it is very difficult to give an accurate estimate for the upper bound \mathcal{K}^* . This then calls for help from accurate numerical methods. Indeed, the methodology and analysis of the vanishing moment method works very well for solving this problem and for estimating \mathcal{K}^* .

Unlike the Monge-Ampère equation considered in the previous section, we have some leeway in defining $F(D^2u, \nabla u, u, x)$. For reasons that will be evident later (cf. Remark 6.8), we set

$$(6.30) \quad F(D^2u, \nabla u, u, x) = -\frac{\det(D^2u)}{(1 + |\nabla u|^2)^{\frac{n+2}{2}}} + \mathcal{K},$$

and therefore,

$$\begin{aligned} F'[v](w) &= -\frac{\text{cof}(D^2v) : D^2w}{(1 + |\nabla v|^2)^{\frac{n+2}{2}}} + (n+2) \frac{\det(D^2v) \nabla v \cdot \nabla w}{(1 + |\nabla v|^2)^{\frac{n+4}{2}}}, \\ F'[\mu, v](\kappa, w) &= -\frac{\text{cof}(\mu) : \kappa}{(1 + |\nabla v|^2)^{\frac{n+2}{2}}} + (n+2) \frac{\det(\mu) \nabla v \cdot \nabla w}{(1 + |\nabla v|^2)^{\frac{n+4}{2}}}. \end{aligned}$$

Therefore, the vanishing moment approximation (2.9)–(2.11)₁ is

$$(6.31) \quad -\varepsilon \Delta^2 u^\varepsilon + \frac{\det(D^2u^\varepsilon)}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}} = \mathcal{K} \quad \text{in } \Omega,$$

$$(6.32) \quad u^\varepsilon = g \quad \text{on } \partial\Omega,$$

$$(6.33) \quad \Delta u^\varepsilon = \varepsilon \quad \text{on } \partial\Omega,$$

and the linearization of

$$G_\varepsilon(u^\varepsilon) = \varepsilon \Delta^2 u^\varepsilon - \frac{\det(D^2u^\varepsilon)}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}} + \mathcal{K}$$

at the solution u^ε is

$$G'_\varepsilon[u^\varepsilon](v) = \varepsilon \Delta^2 v - \frac{\Phi^\varepsilon : D^2v}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}} + (n+2) \frac{\det(D^2u^\varepsilon) \nabla u^\varepsilon \cdot \nabla v}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+4}{2}}},$$

where Φ^ε denotes the cofactor matrix of D^2u^ε .

Numerical tests indicate that there exists a unique strictly convex solution to (6.31)–(6.33) with $\varepsilon > 0$ (cf. Subsection 6.2.3, and [37, 61]). For the continuation of this section, we assume that there exists a unique strictly convex solution to (6.31)–(6.33). Furthermore, since the high-order terms in the equation of prescribed Gauss curvature are the same as the Monge-Ampère equation, we expect that the a priori bounds (6.9)–(6.10) hold for the solution u^ε of the vanishing moment approximation (6.31)–(6.33).

Before stating the finite element methods for (6.31)–(6.33) and applying the analysis of Chapters 4 and 5 to these methods, we first prove the following identity.

Lemma 6.6. *For all $v, w \in H_0^1(\Omega)$*

$$(6.34) \quad \langle F'[u^\varepsilon](v), w \rangle = \left(\frac{\Phi^\varepsilon \nabla v}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}}, \nabla w \right).$$

PROOF. Integrating by parts, we have

$$\begin{aligned} \langle F'[u^\varepsilon](v), w \rangle &= \left(\frac{\Phi^\varepsilon \nabla v}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}}, \nabla w \right) - \frac{n+2}{2} \left(\frac{\Phi^\varepsilon \nabla v \cdot \nabla(|\nabla u^\varepsilon|^2)}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+4}{2}}}, w \right) \\ &\quad + (n+2) \left(\frac{\det(D^2u^\varepsilon) \nabla u^\varepsilon \cdot \nabla v}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+4}{2}}}, w \right). \end{aligned}$$

Noting that $\Phi^\varepsilon D^2u^\varepsilon = \det(D^2u^\varepsilon) I_{n \times n}$, we conclude

$$\Phi^\varepsilon \nabla v \cdot \nabla(|\nabla u^\varepsilon|^2) = \Phi^\varepsilon \nabla v \cdot 2D^2u^\varepsilon \nabla u^\varepsilon = 2 \det(D^2u^\varepsilon) (\nabla u^\varepsilon \cdot \nabla v).$$

From this identity, (6.34) immediately follows. \square

Since u^ε is strictly convex, we arrive at the following corollary.

Corollary 6.7. *There exists a constant $C > 0$ such that*

$$(6.35) \quad \langle F'[u^\varepsilon](w), w \rangle \geq C \|w\|_{H^1}^2 \quad \forall w \in H_0^1(\Omega).$$

Remark 6.8. It is now obvious why we choose (6.30) as the definition of F opposed to the following choice:

$$(6.36) \quad F(D^2u, \nabla u, u, x) = -\det(D^2u) + \mathcal{K}(1 + |\nabla u|^2)^{\frac{n+2}{2}}.$$

Indeed, if we chose (6.36) instead of (6.30) then

$$F'[u^\varepsilon](w) = -\Phi^\varepsilon : D^2w + \mathcal{K}(n+2)(1 + |\nabla u^\varepsilon|^2)^{\frac{n}{2}} \nabla u^\varepsilon \cdot \nabla w,$$

and a simple calculation shows

$$\begin{aligned} \langle F'[u^\varepsilon](w), w \rangle &= (\Phi^\varepsilon \nabla w, \nabla w) - \frac{\mathcal{K}(n+2)}{2} \left((1 + |\nabla u^\varepsilon|^2)^{\frac{n}{2}} \Delta u^\varepsilon \right. \\ &\quad \left. + n(1 + |\nabla u^\varepsilon|^2)^{\frac{n-2}{2}} \tilde{\Delta}_\infty u^\varepsilon, w^2 \right) \quad \forall w \in H_0^1(\Omega), \end{aligned}$$

where $\tilde{\Delta}_\infty u^\varepsilon := D^2u^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon$.¹ Since u^ε is strictly convex, both Δu^ε and $\tilde{\Delta}_\infty u^\varepsilon$ are positive terms, and therefore, the linearization of this choice of F is not coercive.

¹ Throughout this chapter we define $\tilde{\Delta}_\infty v := D^2v \nabla v \cdot \nabla v$ and $\Delta_\infty v := \frac{D^2v \nabla v \cdot \nabla v}{|\nabla v|^2}$. We note that both operators are referred to the infinity-Laplacian in the literature [3, 34]. We shall use the latter definition in Section 6.3.

Nevertheless, the above choice is also valid since it is easy to check that $F'[u^\varepsilon]$ satisfies the following Gårding inequality

$$\langle F'[u^\varepsilon](w), w \rangle \geq \tilde{C}_1 \|w\|_{H^1}^2 - \tilde{C}_0 \|w\|_{L^2}^2 \quad \forall w \in H_0^1(\Omega),$$

for some positive constants $\tilde{C}_0 = \tilde{C}_0(\varepsilon)$, $\tilde{C}_1 = \tilde{C}_1(\varepsilon)$. Here $\langle \cdot, \cdot \rangle$ denotes the dual pairing between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. In addition, other conditions of Assumption (A) also can be verified. We refer the reader to [61] for a detailed exposition.

6.2.1. Conforming finite element methods for the equation of prescribed Gauss curvature. The finite element method for (6.31)–(6.33) is to find $u_h^\varepsilon \in V_g^h$ such that for any $v_h \in V_0^h$

$$(6.37) \quad -\varepsilon(\Delta u_h^\varepsilon, \Delta v_h) + \left(\frac{\det(D^2 u_h^\varepsilon)}{(1 + |\nabla u_h^\varepsilon|^2)^{\frac{n+2}{2}}}, v_h \right) = \mathcal{K}(1, v_h) - \left\langle \varepsilon^2, \frac{\partial v_h}{\partial \nu} \right\rangle_{\partial \Omega}.$$

The goal of this section is to apply the abstract framework of Chapter 4 toward the finite element method (6.37). Specifically, our goal is to show that assumptions [A1]–[A5] hold, and as a consequence, we will obtain existence and uniqueness of a solution to (6.37), as well as optimal order estimates for the error $u^\varepsilon - u_h^\varepsilon$. We also pay close attention on the constants C_i and $L(h)$ and how they depend on the parameter ε . We summarize our results in the following theorem.

Theorem 6.9. *Let $u^\varepsilon \in H^s(\Omega)$ be the solution to (6.31)–(6.33) with $s \geq 3$ when $n = 2$ and $s > 3$ when $n = 3$. Then for $h \leq h_2$, there exists a unique solution to (6.37). Furthermore, there exists positive constants C_7 , C_8 such that*

$$(6.38) \quad \|u^\varepsilon - u_h^\varepsilon\|_{H^2} \leq C_7 h^{\ell-2} \|u^\varepsilon\|_{H^\ell},$$

$$(6.39) \quad \|u^\varepsilon - u_h^\varepsilon\|_{L^2} \leq C_8 \left(\varepsilon^{-\frac{1}{2}} h^\ell \|u^\varepsilon\|_{H^\ell} + C_7 L(h) h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right),$$

where

$$\begin{aligned} C_7 &= O(\varepsilon^{\frac{1}{2}(1-2n)}), & C_8 &= O(C_7 \varepsilon^{-(n+2)}), \\ L(h) &= C(\varepsilon^{-\frac{1}{6}(n+2)} + h^{\frac{3}{2}(2-n)}), & \ell &= \min\{s, k+1\}, \end{aligned}$$

and h_2 is chosen such that

$$h_2 \leq C \left(\varepsilon^{-\frac{1}{2}(1+2n)} \|u^\varepsilon\|_{H^\ell} L(h_2) \right)^{\frac{1}{2-\ell}}.$$

PROOF. First, (6.35) implies that

$$(6.40) \quad \langle G'_\varepsilon[u^\varepsilon](v), v \rangle \geq C\varepsilon \|v\|_{H^2}^2 \quad \forall v \in V_0,$$

and it follows that $(G'_\varepsilon[u^\varepsilon])^*$ is an isomorphism from V_0 to V_0^* .

Next, for any $v, w \in V_0$, using (6.9) and a Sobolev inequality, we have

$$\begin{aligned} (6.41) \quad \langle F'[u^\varepsilon](v), w \rangle &= \left(\frac{\Phi^\varepsilon \nabla v}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}}, \nabla w \right) \\ &\leq \|\Phi^\varepsilon\|_{L^{\frac{3}{2}}} \|\nabla v\|_{L^6} \|\nabla w\|_{L^6} \\ &\leq C \|\Phi^\varepsilon\|_{L^{\frac{3}{2}}} \|v\|_{H^2} \|w\|_{H^2}, \end{aligned}$$

and therefore,

$$\|F'[u^\varepsilon]\|_{VV^*} = \sup_{v \in V_0} \sup_{w \in V_0} \frac{\langle F'[u^\varepsilon](v), w \rangle}{\|v\|_{H^2} \|w\|_{H^2}} \leq C \|\Phi^\varepsilon\|_{L^{\frac{3}{2}}}.$$

In view of Remark 4.4 and the estimates

$$\begin{aligned} \left\| \frac{\partial F(u^\varepsilon)}{\partial r_{ij}} \right\|_{L^\infty} &= \left\| \Phi_{ij}^\varepsilon / (1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}} \right\|_{L^\infty} = O(\varepsilon^{-1}), \\ \left\| \frac{\partial F(u^\varepsilon)}{\partial p_i} \right\|_{L^\infty} &= (n+2) \left\| \det(D^2 u^\varepsilon) \frac{\partial u^\varepsilon}{\partial x_i} / (1 + |\nabla u^\varepsilon|^2)^{\frac{n+4}{2}} \right\|_{L^\infty} \\ &\leq C \|\nabla u^\varepsilon\|_{L^\infty} \|\det(D^2 u^\varepsilon)\|_{L^\infty} = O(\varepsilon^{-n}), \end{aligned}$$

we conclude that if $v \in V_0$ is the solution to

$$(6.42) \quad \langle (G'_\varepsilon[u^\varepsilon])^*(v), w \rangle = \langle \varphi, w \rangle \quad \forall w \in V_0,$$

for some $\varphi \in L^2(\Omega)$, then

$$(6.43) \quad \|v\|_{H^3} \leq C\varepsilon^{-2} \|\varphi\|_{L^2} \quad \|v\|_{H^4} \leq C\varepsilon^{-(n+2)} \|\varphi\|_{L^2}.$$

Thus, by (6.40)–(6.43), [A2] holds with

$$(6.44) \quad \begin{aligned} C_0 &\equiv 0, & C_1 &= O(\varepsilon), & C_2 &= O(\varepsilon^{-\frac{1}{2}}), \\ p &= 4, & C_R &= O(\varepsilon^{-(n+2)}), \end{aligned}$$

and therefore (cf. Theorem 4.3)

$$(6.45) \quad \begin{aligned} C_3 &= O(\varepsilon^{-1}), & C_4 &= O(\varepsilon^{-\frac{3}{2}}), \\ C_5 &= O(\varepsilon^{-(4+n)}), & h_0 &= 1. \end{aligned}$$

To confirm [A3]–[A4], we take

$$Y = W^{2, \frac{3(n-1)}{2}}(\Omega), \quad \|\cdot\|_Y = \|\cdot\|_{W^{2, \frac{3(n-1)}{2}}}^{n-1}.$$

We then have the following bound for any $v, z \in V_0$, $y \in Y$:

$$\begin{aligned} \left(\frac{\text{cof}(D^2 y) \nabla v}{(1 + |\nabla y|^2)^{\frac{n+2}{2}}}, \nabla z \right) &\leq \|\text{cof}(D^2 y)\|_{L^{\frac{3}{2}}} \|\nabla v\|_{L^6} \|\nabla z\|_{L^6} \\ &\leq C \|\text{cof}(D^2 y)\|_{L^{\frac{3}{2}}} \|v\|_{H^2} \|z\|_{H^2} \\ &\leq C \|D^2 y\|_{L^{\frac{n-1}{3(n-1)}}}^{n-1} \|v\|_{H^2} \|z\|_{H^2}. \end{aligned}$$

It then follows that

$$\sup_{y \in Y} \frac{\|F'[y]\|_{VV^*}}{\|y\|_Y} \leq C,$$

and thus, [A3]–[A4] holds. We also note from (6.9)–(6.10) that

$$(6.46) \quad \|u^\varepsilon\|_Y = O\left(\varepsilon^{\frac{1}{2}(3-2n)}\right), \quad C_6 = \left(\varepsilon^{\frac{1}{2}(1-2n)}\right).$$

To verify condition [A5], we first make the following calculation for any $w, z \in V_0$, $v_h \in V_g^h$:

$$\begin{aligned}
 (6.47) \quad & \langle (F'[u^\varepsilon] - F'[v_h])(w), z \rangle \\
 &= \left(\frac{\Phi^\varepsilon \nabla w}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}} - \frac{\text{cof}(D^2 v_h) \nabla w}{(1 + |\nabla v_h|^2)^{\frac{n+2}{2}}}, \nabla z \right) \\
 &= \left(\frac{(\Phi^\varepsilon - \text{cof}(D^2 v_h)) \nabla w}{(1 + |\nabla v_h|^2)^{\frac{n+2}{2}}}, \nabla z \right) \\
 &\quad + \left(\frac{\Phi^\varepsilon \nabla w}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}} - \frac{\Phi^\varepsilon \nabla w}{(1 + |\nabla v_h|^2)^{\frac{n+2}{2}}}, \nabla z \right).
 \end{aligned}$$

Bounding the first term in (6.47), we use a Sobolev inequality to conclude

$$(6.48) \quad \left(\frac{(\Phi^\varepsilon - \text{cof}(D^2 v_h)) \nabla w}{(1 + |\nabla v_h|^2)^{\frac{n+2}{2}}}, \nabla z \right) \leq C \|\Phi^\varepsilon - \text{cof}(D^2 v_h)\|_{L^{\frac{3}{2}}} \|w\|_{H^2} \|z\|_{H^2}.$$

To bound the second term in (6.47), we first use the mean value theorem

$$\begin{aligned}
 & (1 + |\nabla u^\varepsilon|^2)^{-\frac{n+2}{2}} - (1 + |\nabla v_h|^2)^{-\frac{n+2}{2}} \\
 &= -(n+2)(1 + |\nabla y_h|^2)^{-\frac{n+4}{2}} \nabla y_h \cdot \nabla(u^\varepsilon - v_h),
 \end{aligned}$$

where $y_h = u^\varepsilon + \gamma v_h$ for some $\gamma \in [0, 1]$. Therefore, for any $\delta \in (0, 1)$ and $v_h \in V_g^h$ with $\|\mathcal{I}^h u^\varepsilon - v_h\|_{H^2} \leq \delta$

$$\begin{aligned}
 & \left(\frac{\Phi^\varepsilon \nabla w}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}} - \frac{\Phi^\varepsilon \nabla w}{(1 + |\nabla v_h|^2)^{\frac{n+2}{2}}}, \nabla z \right) \\
 & \leq C \|\nabla y_h\|_{L^6} \|\nabla(u^\varepsilon - v_h)\|_{L^6} \|\Phi^\varepsilon\|_{L^3} \|\nabla w\|_{L^6} \|\nabla z\|_{L^6} \\
 & \leq C(\|u\|_{W^{1,\infty}} + \delta) \|u^\varepsilon - v_h\|_{H^2} \|\Phi^\varepsilon\|_{L^3} \|w\|_{H^2} \|z\|_{H^2} \\
 & \leq C\varepsilon^{-\frac{2}{3}} \|u^\varepsilon - v_h\|_{H^2} \|w\|_{H^2} \|z\|_{H^2}.
 \end{aligned}$$

It then follows from this calculation and (6.48) that in the two-dimensional case

$$\|F'[u^\varepsilon] - F'[v_h]\|_{V_{V^*}} \leq C\varepsilon^{-\frac{2}{3}} \|u^\varepsilon - v_h\|_{H^2} = L(h) \|u^\varepsilon - v_h\|_{H^2},$$

that is, condition [A5] holds with $L(h) = C\varepsilon^{-\frac{2}{3}}$.

In the three-dimensional setting, using arguments similar to those for the Monge-Ampère equation, we have

$$\|F'[u^\varepsilon] - F'[v_h]\|_{V_{V^*}} \leq C(\varepsilon^{-\frac{5}{6}} + h^{-1}) \|u^\varepsilon - v_h\|_{H^2} = L(h) \|u^\varepsilon - v_h\|_{H^2},$$

and therefore [A5] holds with $L(h) = C(\varepsilon^{-\frac{5}{6}} + h^{-1})$.

Gathering up these results, and applying Theorem 4.7 with estimates (6.44)–(6.46), we conclude that there exists a unique solution to the finite element method (6.37) and that the error estimates (6.38)–(6.39) hold. \square

6.2.2. Mixed finite element methods for the equation of prescribed Gauss curvature. The mixed finite element method for (6.28)–(6.29) is defined as follows: find $(\sigma_h^\varepsilon, u_h^\varepsilon) \in W_\varepsilon^h \times Q_g^h$ such that

$$(6.49) \quad (\sigma_h^\varepsilon, \kappa_h) + b(\kappa_h, u_h^\varepsilon) = G(\kappa_h) \quad \forall \kappa_h \in W_0^h,$$

$$(6.50) \quad b(\sigma_h^\varepsilon, z_h) - \varepsilon^{-1} c(\sigma_h^\varepsilon, u_h^\varepsilon, z_h) = 0 \quad \forall z_h \in Q_0^h,$$

where

$$b(\kappa_h, u_h^\varepsilon) = (\operatorname{div}(\kappa_h), \nabla u_h^\varepsilon),$$

$$c(\sigma_h^\varepsilon, u_h^\varepsilon, z_h) = \left(\mathcal{K} - \frac{\det(\sigma_h^\varepsilon)}{(1 + |\nabla u_h^\varepsilon|^2)^{\frac{n+2}{2}}}, z_h \right),$$

and $G(\kappa_h)$ is defined by (5.7).

In this section, we apply the results of Chapter 5 to the mixed finite element method (6.49)–(6.50). Namely, we verify that conditions [B1]–[B6] hold, and from these results, we obtain existence and uniqueness of a solution to (6.49)–(6.50) as well as its error estimates. We summarize our findings in the following theorem.

Theorem 6.10. *Let $u^\varepsilon \in H^s(\Omega)$ be the solution to (6.31)–(6.33) with $s > 3$ when $n = 2$ and $s > 5$ when $n = 3$. Suppose $k \geq 3$ when $n = 2$ and $k \geq 5$ when $n = 3$. Then for $h \leq h_2$, there exists a unique solution $(\sigma_h^\varepsilon, u_h^\varepsilon) \in W_\varepsilon^h \times Q_g^h$ to (6.49)–(6.49). Furthermore, there hold the following error estimates:*

$$(6.51) \quad \|(\sigma^\varepsilon - \sigma_h^\varepsilon, u^\varepsilon - u_h^\varepsilon)\|_\varepsilon \leq K_8 h^{\ell-2} \|u^\varepsilon\|_{H^\ell},$$

$$(6.52) \quad \|u^\varepsilon - u_h^\varepsilon\|_{H^1} \leq K_{R_1} \left(K_9 h^{\ell-1} \|u^\varepsilon\|_{H^\ell} + K_8^2 R(h) h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right),$$

where

$$\|(\mu, v)\|_\varepsilon = h \|\mu\|_{H^1} + \|\mu\|_{L^2} + \varepsilon^{-\frac{1}{2}} \|v\|_{H^1},$$

$$K_8 = O(\varepsilon^{\frac{1}{4}(28-19n)}), \quad K_9 = O(\varepsilon^{\frac{76-49n}{12}}),$$

$$R(h) = C \begin{cases} |\log h|(h^{-1} + \varepsilon^{-2}) & n = 2, \\ \varepsilon^{-1} h^{-1} + \varepsilon^{-3} + h^{-3} & n = 3, \end{cases}$$

$$\ell = \min\{s, k+1\},$$

and h_2 is chosen such that

$$h_2 = C \min \left\{ \varepsilon^{\frac{7+4n}{6}}, \left(\varepsilon^{\frac{1}{4}(26-19n)} R(h_2) \|u^\varepsilon\|_{H^\ell} \right)^{\frac{1}{1-\ell}}, \left(\varepsilon^{-\frac{1}{2}} R(h_2) \|u^\varepsilon\|_{H^\ell} \right)^{\frac{1}{2-\ell}} \right\}.$$

PROOF. First, using the same arguments as those used to show assumption [A2] in Theorem 6.9, we can conclude that [B2] holds with

$$(6.53) \quad \begin{aligned} K_0 &\equiv 0, & K_1 &= O(1), & K_2 &= O(\varepsilon^{-1}), \\ p &= 4, & K_{R_0} &= O(\varepsilon^{-(n+2)}), & K_{R_1} &= O(\varepsilon^{-2}), \end{aligned}$$

and therefore, (cf. Theorem 5.6 and Lemma 5.9)

$$(6.54) \quad K_4 = O(\varepsilon^{-\frac{3}{2}}), \quad K_5 = O(\varepsilon^{-\frac{1}{2}(2n+5)}), \quad K_7 = O(\varepsilon^{-\frac{1}{2}}).$$

We now turn our attention to condition [B3]. To show that this condition holds, we set $(n = 2, 3)$

$$X = \left[L^{n(n+\varepsilon(3-n))}(\Omega) \right]^{n \times n}, \quad Y = W^{1,\infty}(\Omega),$$

$$\|(\omega, y)\|_{X \times Y} = \|\omega\|_{L^{(n-1)(n+\varepsilon(3-n))}}^{n-1} + \|\omega\|_{L^{n(n+\varepsilon(3-n))}}^n \|\nabla y\|_{L^\infty} \quad \forall \omega \in X, y \in Y.$$

Then for any $\omega \in X$, $y \in Y$, $\chi \in W$, and $v \in Q$, $z \in Q_0$, we have

$$\begin{aligned} \langle F'[\omega, y](\chi, v), z \rangle &= - \left(\frac{\text{cof}(\omega) : \chi}{(1 + |\nabla y|^2)^{\frac{n+2}{2}}}, z \right) + (n+2) \left(\frac{\det(\omega) \nabla y \cdot \nabla v}{(1 + |\nabla y|^2)^{\frac{n+4}{2}}}, z \right) \\ &\leq C \left(\|\text{cof}(\omega)\|_{L^{n+\varepsilon(3-n)}} \|\chi\|_{L^2} \|z\|_{H^1} + \|\det(\omega)\|_{L^{n+\varepsilon(3-n)}} \|\nabla y\|_{L^\infty} \|\nabla v\|_{L^2} \|z\|_{H^1} \right) \\ &\leq C \left(\|\omega\|_{L^{(n-1)(n+\varepsilon(3-n))}}^{n-1} \|\chi\|_{L^2} + \|\omega\|_{L^{n(n+\varepsilon(3-n))}}^n \|\nabla y\|_{L^\infty} \|v\|_{H^1} \right) \|z\|_{H^1} \\ &\leq C \left(\|\omega\|_{L^{(n-1)(n+\varepsilon(3-n))}}^{n-1} + \|\omega\|_{L^{n(n+\varepsilon(3-n))}}^n \|\nabla y\|_{L^\infty} \right) (\|\chi\|_{L^2} + \|v\|_{H^1}) \|z\|_{H^1}. \end{aligned}$$

It follows from this calculation that

$$\|F'[\omega, y](\chi, v)\|_{H^{-1}} \leq C \|(\omega, y)\|_{X \times Y} (\|\chi\|_{L^2} + \|v\|_{H^1}),$$

and therefore condition [B3] holds.

To confirm [B4], we use (6.25) to conclude that if $\sigma^\varepsilon \in [H^{s-2}(\Omega)]^{n \times n}$, then for any $\gamma \in [0, 1]$ and $\ell \in [3, \min\{s, k+1\}]$

$$\begin{aligned} &\left\| \left(\Pi^h \sigma^\varepsilon - \gamma \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon - \gamma u^\varepsilon \right) \right\|_{X \times Y} \\ &= \left\| \Pi^h \sigma^\varepsilon - \gamma \sigma^\varepsilon \right\|_{L^{(n-1)(n+\varepsilon(3-n))}}^{n-1} + \left\| \Pi^h \sigma^\varepsilon - \gamma \sigma^\varepsilon \right\|_{L^{n(n+\varepsilon(3-n))}}^n \left\| \nabla \mathcal{I}^h u^\varepsilon - \gamma \nabla u^\varepsilon \right\|_{L^\infty} \\ &\leq C \left(\left\| \Pi^h \sigma^\varepsilon \right\|_{L^{n(n+\varepsilon(3-n))}}^n + \left\| \sigma^\varepsilon \right\|_{L^{n(n+\varepsilon(3-n))}}^n \right) \\ &\leq C \left(h^{\ell - \frac{5}{3} - \frac{n}{2}} \|\sigma^\varepsilon\|_{H^{\ell-2}} + \|\sigma^\varepsilon\|_{L^{n(n+\varepsilon(3-n))}} \right)^n, \end{aligned}$$

For the two-dimensional case, we set $\ell = 3$ and use (6.10) to get

$$\left\| \left(\Pi^h \sigma^\varepsilon - \gamma \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon - \gamma u^\varepsilon \right) \right\|_{X \times Y} \leq C \left(h^{\frac{1}{3}} \|\sigma^\varepsilon\|_{H^1} + \|\sigma^\varepsilon\|_{L^{4+2\varepsilon}} \right)^2 = O(\varepsilon^{-2}).$$

For the three-dimensional case, we set $\ell = \frac{19}{6}$ and use (6.12) and (6.10) to conclude

$$\left\| \left(\Pi^h \sigma^\varepsilon - \gamma \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon - \gamma u^\varepsilon \right) \right\|_{X \times Y} \leq C \left(\|\sigma^\varepsilon\|_{H^{\frac{7}{6}}} + \|\sigma^\varepsilon\|_{L^9} \right)^3 = O(\varepsilon^{-\frac{27}{4}}).$$

Combing these two estimates, we have

$$(6.55) \quad K_3 = O\left(\varepsilon^{\frac{1}{4}(30-19n)}\right), \quad K_6 = O\left(\varepsilon^{\frac{1}{4}(28-19n)}\right).$$

As a first step to confirm [B5], we note that for all $(\mu_h, v_h) \in W_\varepsilon^h \times Q_g^h$ and $(\kappa_h, z_h) \in W^h \times Q^h$, $w_h \in Q^h$

$$\begin{aligned}
 (6.56) \quad & \left\langle (F'[\sigma^\varepsilon, u^\varepsilon] - F'[\mu_h, v_h])(\kappa_h, z_h), w_h \right\rangle \\
 &= \left(\left(\frac{\text{cof}(\mu_h)}{(1 + |\nabla v_h|^2)^{\frac{n+2}{2}}} - \frac{\text{cof}(\sigma^\varepsilon)}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}} \right) : \kappa_h, w_h \right) \\
 &\quad + (n+2) \left(\left(\frac{\det(\sigma^\varepsilon) \nabla u^\varepsilon}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+4}{2}}} - \frac{\det(\mu_h) \nabla v_h}{(1 + |\nabla v_h|^2)^{\frac{n+4}{2}}} \right) \cdot \nabla z_h, w_h \right).
 \end{aligned}$$

To bound the first term in (6.56), we use the mean value theorem to conclude

$$\begin{aligned}
 & \left(\left(\frac{\text{cof}(\mu_h)}{(1 + |\nabla v_h|^2)^{\frac{n+2}{2}}} - \frac{\text{cof}(\sigma^\varepsilon)}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}} \right) : \kappa_h, w_h \right) \\
 &= \left(\frac{(\text{cof}(\mu_h) - \text{cof}(\sigma^\varepsilon)) : \kappa_h}{(1 + |\nabla v_h|^2)^{\frac{n+2}{2}}}, w_h \right) + \left(\frac{\text{cof}(\sigma^\varepsilon) : \kappa_h}{(1 + |\nabla v_h|^2)^{\frac{n+2}{2}}}, w_h \right) \\
 &\quad - \left(\frac{\text{cof}(\sigma^\varepsilon) : \kappa_h}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}}, w_h \right) \\
 &= \left(\frac{(\text{cof}(\mu_h) - \text{cof}(\sigma^\varepsilon)) : \kappa_h}{(1 + |\nabla v_h|^2)^{\frac{n+2}{2}}}, w_h \right) - (n+2) \left(\frac{\nabla y_h \cdot \nabla(v_h - u^\varepsilon) \text{cof}(\sigma^\varepsilon) : \kappa_h}{(1 + |\nabla y_h|^2)^{\frac{n+4}{2}}}, w_h \right),
 \end{aligned}$$

where $y_h = v_h + \gamma u^\varepsilon$ for some $\gamma \in [0, 1]$. Therefore, by (6.9) and the inverse inequality, we have

$$\begin{aligned}
 (6.57) \quad & \left(\left(\frac{\text{cof}(\mu_h)}{(1 + |\nabla v_h|^2)^{\frac{n+2}{2}}} - \frac{\text{cof}(\sigma^\varepsilon)}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}} \right) : \kappa_h, w_h \right) \\
 &\leq C \left(\|\text{cof}(\mu_h) - \text{cof}(\sigma^\varepsilon)\|_{L^2} + \|\nabla y_h\|_{L^\infty} \|u^\varepsilon - v_h\|_{H^1} \|\text{cof}(\sigma^\varepsilon)\|_{L^\infty} \right) \\
 &\quad \times \|\kappa_h\|_{L^2} \|w_h\|_{L^\infty} \\
 &\leq C |\log h|^{\frac{3-n}{2}} h^{1-\frac{n}{2}} \left(\|\text{cof}(\mu_h) - \text{cof}(\sigma^\varepsilon)\|_{L^2} + \varepsilon^{-1} \|\nabla y_h\|_{L^\infty} \|u^\varepsilon - v_h\|_{H^1} \right) \\
 &\quad \times \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1}.
 \end{aligned}$$

If $v_h \in Q_g^h$ with $\|\mathcal{I}^h u^\varepsilon - v_h\|_{H^1} \leq \delta \in (0, 1)$, then by the inverse inequality

$$\begin{aligned}
 \|\nabla y_h\|_{L^\infty} &\leq \|\nabla u^\varepsilon\|_{L^\infty} + \|\nabla \mathcal{I}^h u^\varepsilon\|_{L^\infty} + |\log h|^{\frac{3-n}{2}} h^{1-\frac{n}{2}} \|\nabla(\mathcal{I}^h u^\varepsilon - v_h)\|_{H^1} \\
 &\leq C |\log h|^{\frac{3-n}{2}} h^{1-\frac{n}{2}}.
 \end{aligned}$$

Furthermore, from the mixed finite element analysis for the Monge-Ampère equation, we have

$$\|\text{cof}(\sigma^\varepsilon) - \text{cof}(\mu_h)\|_{L^2} \leq C(\varepsilon^{(2-n)} + h^{\frac{3}{2}(2-n)}) \|\sigma^\varepsilon - \mu_h\|_{L^2}.$$

Using these two inequalities in (6.57), we arrive at

$$\begin{aligned}
(6.58) \quad & \left(\left(\frac{\text{cof}(\mu_h)}{(1 + |\nabla v_h|^2)^{\frac{n+2}{2}}} - \frac{\text{cof}(\sigma^\varepsilon)}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}} \right) : \kappa_h, w_h \right) \\
& \leq C |\log h|^{\frac{3-n}{2}} h^{1-\frac{n}{2}} \left((\varepsilon^{(2-n)} + h^{\frac{3}{2}(2-n)}) \|\sigma^\varepsilon - \mu_h\|_{L^2} \right. \\
& \quad \left. + \varepsilon^{-1} |\log h|^{\frac{3-n}{2}} h^{1-\frac{n}{2}} \|u^\varepsilon - v_h\|_{H^1} \right) \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1} \\
& \leq C (\varepsilon^{-1} |\log h|)^{3-n} (\varepsilon^{-1} h^{-\frac{1}{2}} + h^{-2})^{n-2} (\|\sigma^\varepsilon - \mu_h\|_{L^2} + \|u^\varepsilon - v_h\|_{H^1}).
\end{aligned}$$

Using a similar strategy to bound the second term in (6.56), we add and subtract terms and use the mean value theorem to conclude

$$\begin{aligned}
(6.59) \quad & \left(\left(\frac{\det(\sigma^\varepsilon) \nabla u^\varepsilon}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+4}{2}}} - \frac{\det(\mu_h) \nabla v_h}{(1 + |\nabla v_h|^2)^{\frac{n+4}{2}}} \right) \cdot \nabla z_h, w_h \right) \\
& = \left(\frac{(\det(\sigma^\varepsilon) - \det(\mu_h)) \nabla u^\varepsilon \cdot \nabla z_h}{(1 + |\nabla v_h|^2)^{\frac{n+4}{2}}} + \frac{\det(\mu_h) (\nabla u^\varepsilon - \nabla v_h) \cdot \nabla z_h}{(1 + |\nabla v_h|^2)^{\frac{n+4}{2}}}, w_h \right) \\
& \quad + \left(\frac{\det(\sigma^\varepsilon) \nabla u^\varepsilon \cdot \nabla z_h}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+4}{2}}} - \frac{\det(\sigma^\varepsilon) \nabla u^\varepsilon \cdot \nabla z_h}{(1 + |\nabla v_h|^2)^{\frac{n+4}{2}}}, w_h \right) \\
& = \left(\frac{(\text{cof}(\xi_h) : (\sigma^\varepsilon - \mu_h)) \nabla u^\varepsilon \cdot \nabla z_h}{(1 + |\nabla v_h|^2)^{\frac{n+4}{2}}} + \frac{\det(\mu_h) (\nabla u^\varepsilon - \nabla v_h) \cdot \nabla z_h}{(1 + |\nabla v_h|^2)^{\frac{n+4}{2}}}, w_h \right) \\
& \quad - (n+4) \left(\frac{\det(\sigma^\varepsilon) (\nabla u^\varepsilon \cdot \nabla z_h) (\nabla x_h \cdot \nabla (u^\varepsilon - v_h))}{(1 + |\nabla x_h|^2)^{\frac{n+6}{2}}}, w_h \right),
\end{aligned}$$

where $\xi_h = \sigma^\varepsilon + \gamma_1 \mu_h$, $x_h = u^\varepsilon + \gamma_2 v_h$ for some $\gamma_1, \gamma_2 \in [0, 1]$. Bounding the first term in (6.59), we use the inverse inequality to conclude

$$\begin{aligned}
& \left(\frac{(\text{cof}(\xi_h) : (\sigma^\varepsilon - \mu_h)) \nabla u^\varepsilon \cdot \nabla z_h}{(1 + |\nabla v_h|^2)^{\frac{n+4}{2}}}, w_h \right) \\
& \leq C \|\text{cof}(\xi_h)\|_{L^2} \|\sigma^\varepsilon - \mu_h\|_{L^2} \|\nabla u^\varepsilon\|_{L^\infty} \|\nabla z_h\|_{L^\infty} \|w_h\|_{L^\infty} \\
& \leq C |\log h|^{\frac{3-n}{2}} h^{1-n} \|\text{cof}(\xi_h)\|_{L^2} \|\sigma^\varepsilon - \mu_h\|_{L^2} \|z_h\|_{H^1} \|w_h\|_{H^1}.
\end{aligned}$$

If $\|\Pi^h \sigma^\varepsilon - \mu_h\|_{L^2} \leq \delta \in (0, 1)$, then by (6.10) and the inverse inequality

$$\begin{aligned}
\|\text{cof}(\xi_h)\|_{L^2} & \leq \|\xi_h\|_{L^{2(n-1)}}^{n-1} \leq C \|\sigma^\varepsilon\|_{L^{2(n-1)}}^{n-1} + \|\Pi^h \sigma^\varepsilon - \mu_h\|_{L^{2(n-1)}}^{n-1} \\
& \leq C (\varepsilon^{\frac{3-2n}{2}} + h^{\lfloor \frac{n}{2(n-1)} - \frac{n}{2} \rfloor (n-1)}) \|\Pi^h \sigma^\varepsilon - \mu_h\|_{L^2} \\
& = O \left(\varepsilon^{\frac{3-2n}{2}} + h^{\frac{n(n-2)}{2}} \right) = O \left(\varepsilon^{\frac{3-2n}{2}} + h^{\frac{3}{2}(2-n)} \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(6.60) \quad & \left(\frac{(\text{cof}(\xi_h) : (\sigma^\varepsilon - \mu_h)) \nabla u^\varepsilon \cdot \nabla z_h}{(1 + |\nabla v_h|^2)^{\frac{n+4}{2}}}, w_h \right) \\
& \leq C |\log h|^{\frac{3-n}{2}} h^{1-n} (\varepsilon^{\frac{3-2n}{2}} + h^{\frac{3}{2}(2-n)}) \|\sigma^\varepsilon - \mu_h\|_{L^2} \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1}.
\end{aligned}$$

Bounding the second term in (6.59), we have

$$\begin{aligned}
& \left(\frac{\det(\mu_h)(\nabla u^\varepsilon - \nabla v_h) \cdot \nabla z_h}{(1 + |\nabla v_h|^2)^{\frac{n+4}{2}}}, w_h \right) \\
& \leq \|\det(\mu_h)\|_{L^2} \|\nabla u^\varepsilon - \nabla v_h\|_{L^2} \|\nabla z_h\|_{L^\infty} \|w_h\|_{L^\infty} \\
& \leq C |\log h|^{3-n} h^{2-n} \|\det(\mu_h)\|_{L^2} \|u^\varepsilon - v_h\|_{H^1} \|z_h\|_{H^1} \|w_h\|_{H^1}.
\end{aligned}$$

If $\|\Pi^h \sigma^\varepsilon - \mu_h\|_{L^2} \leq \delta \in (0, 1)$, then by (6.10) and the inverse inequality,

$$\begin{aligned}
\|\det(\mu_h)\|_{L^2} & \leq C \|\mu_h\|_{L^{2n}}^n \leq C (\|\sigma^\varepsilon\|_{L^{2n}}^n + \|\Pi^h \sigma^\varepsilon - \mu_h\|_{L^{2n}}^n) \\
& = O(\varepsilon^{\frac{1-2n}{2}} + h^{\frac{n}{2}(1-n)}) = O(\varepsilon^{\frac{1-2n}{2}} + h^{-2n+3}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(6.61) \quad & \left(\frac{\det(\mu_h)(\nabla u^\varepsilon - \nabla v_h) \cdot \nabla z_h}{(1 + |\nabla v_h|^2)^{\frac{n+4}{2}}}, w_h \right) \\
& \leq C |\log h|^{3-n} h^{2-n} (\varepsilon^{\frac{1-2n}{2}} + h^{-2n+3}) \|u^\varepsilon - v_h\|_{H^1} \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1}.
\end{aligned}$$

Next, using similar arguments as above, we bound the third term in (6.59) as follows:

$$\begin{aligned}
(6.62) \quad & \left(\frac{\det(\sigma^\varepsilon)(\nabla u^\varepsilon \cdot \nabla z_h)(\nabla x_h \cdot \nabla(u^\varepsilon - v_h))}{(1 + |\nabla x_h|^2)^{\frac{n+6}{2}}}, w_h \right) \\
& \leq \|\det(\sigma^\varepsilon)\|_{L^\infty} \|\nabla u^\varepsilon\|_{L^\infty} \|\nabla z_h\|_{L^2} \|\nabla x_h\|_{L^\infty} \|\nabla(u^\varepsilon - v_h)\|_{L^2} \|w_h\|_{L^\infty} \\
& \leq C |\log h|^{\frac{3-n}{2}} h^{1-\frac{n}{2}} \|\sigma^\varepsilon\|_{L^\infty}^n \|\nabla x_h\|_{L^\infty} \|u^\varepsilon - v_h\|_{H^1} \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1} \\
& \leq C |\log h|^{\frac{3-n}{2}} h^{1-\frac{n}{2}} \varepsilon^{-n} \|\nabla x_h\|_{L^\infty} \|u^\varepsilon - v_h\|_{H^1} \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1} \\
& \leq C |\log h|^{3-n} h^{2-n} \varepsilon^{-n} \|u^\varepsilon - v_h\|_{H^1} \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1}.
\end{aligned}$$

Applying the bounds (6.60)–(6.62) to (6.59), we have

$$\begin{aligned}
& \left(\left(\frac{\det(\sigma^\varepsilon) \nabla u^\varepsilon}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+4}{2}}} - \frac{\det(\mu_h) \nabla v_h}{(1 + |\nabla v_h|^2)^{\frac{n+4}{2}}} \right) \cdot \nabla z_h, w_h \right) \\
& \leq C \left(|\log h|^{\frac{3-n}{2}} h^{1-n} (\varepsilon^{\frac{3-2n}{2}} + h^{\frac{3}{2}(2-n)}) \right. \\
& \quad \left. + |\log h|^{3-n} h^{2-n} (\varepsilon^{\frac{1-2n}{2}} + h^{-2n+3}) + |\log h|^{3-n} h^{2-n} \varepsilon^{-n} \right) \\
& \quad \times \left(\|\sigma^\varepsilon - \mu_h\|_{L^2} + \|u^\varepsilon - v_h\|_{H^1} \right) \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1} \\
& \leq C |\log h|^{3-n} (\varepsilon^{-n} + h^{-2n+3}) (\|\sigma^\varepsilon - \mu_h\|_{L^2} + \|u^\varepsilon - v_h\|_{H^1}) \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1}.
\end{aligned}$$

Finally, we combine this last inequality with (6.56)–(6.59) to get

$$\begin{aligned}
& \left\langle (F'[\sigma^\varepsilon, u^\varepsilon] - F'[\mu_h, v_h])(\kappa_h, z_h), w_h \right\rangle \\
& \leq C \left(\varepsilon^{-1} |\log h|^{3-n} h^{2-n} + |\log h|^{3-n} (\varepsilon^{-n} + h^{-2n+3}) \right) \\
& \quad \times (\|\sigma^\varepsilon - \mu_h\|_{L^2} + \|u^\varepsilon - v_h\|_{H^1}) \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1} \\
& \leq C |\log h|^{3-n} (\varepsilon^{-1} h^{2-n} + \varepsilon^{-n} + h^{-2n+3}) \\
& \quad \times (\|\sigma^\varepsilon - \mu_h\|_{L^2} + \|u^\varepsilon - v_h\|_{H^1}) \|(\kappa_h, z_h)\|_\varepsilon \|w_h\|_{H^1}.
\end{aligned}$$

Therefore, condition [B5] holds with $R(h) = C|\log h|(\varepsilon^{-2} + h^{-1})$ in the two-dimensional case and $R(h) = C(\varepsilon^{-1}h^{-1} + \varepsilon^{-3} + h^{-3})$ in the three-dimensional case.

To establish condition [B6], we use similar arguments to that of the mixed finite element analysis of the Monge-Ampère equation to conclude

$$\begin{aligned} \left\| \frac{\partial F}{\partial r_{ij}} \right\|_{L^\infty} &\leq \left\| \frac{\text{cof}(\sigma^\varepsilon)}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}} \right\|_{L^\infty} \leq \|\text{cof}(\sigma^\varepsilon)\|_{L^\infty} = O(\varepsilon^{-1}), \\ \left\| \frac{\partial F}{\partial r_{ij}} \right\|_{W^{1, \frac{6}{5}}} &\leq C \left\| (1 + |\nabla u^\varepsilon|^2)^{-\frac{n+2}{2}} \right\|_{W^{1, \infty}} \|\text{cof}(\sigma^\varepsilon)\|_{W^{1, \frac{6}{5}}} \\ &\leq C \|u^\varepsilon\|_{W^{2, \infty}} \|\text{cof}(\sigma^\varepsilon)\|_{W^{1, \frac{6}{5}}} = O\left(\varepsilon^{-\frac{2}{3}(1+n)}\right). \end{aligned}$$

Therefore by Proposition 5.4, condition [B6] holds with

$$\alpha = 1, \quad K_G = C\varepsilon^{-\frac{2}{3}(1+n)}.$$

Wrapping things up, we apply Theorem 5.10 and 5.11 to obtain existence and uniqueness of a solution $(\sigma_h^\varepsilon, u_h^\varepsilon)$ to the mixed finite element method (6.49)–(6.50). The error estimates (6.51)–(6.52) also follow from these results and by the definitions

$$K_8 = CK_7, \quad K_9 = K_8 \max\{K_2, K_G\}.$$

□

6.2.3. Numerical experiments and rates of convergence. In this section, we provide several two-dimensional numerical experiments to gauge the efficiency of the finite element methods developed in the previous two subsections.

Test 6.2.1. In this test, we fix $h = 0.01$ in order to study the behavior of u^ε . Notably, we are interested whether $\|u - u^\varepsilon\| \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. To this end, we solve the following problem: find $u_h^\varepsilon \in V_h^h$ such that²

$$-\varepsilon(\Delta u_h^\varepsilon, \Delta v_h) + \left(\frac{\det(D^2 u_h^\varepsilon)}{(1 + |\nabla u^\varepsilon|^2)^{\frac{n+2}{2}}}, v_h \right) = (\mathcal{K}f, v_h) - \left\langle \varepsilon^2, \frac{\partial v_h}{\partial \nu} \right\rangle_{\partial \Omega}.$$

Here, we take V^h to be the Argyris finite element space [22] of degree $k = 5$ and set $\Omega = (0, 1)^2$. We use the following test function and parameters:

$$\begin{aligned} \text{(a)} \quad u &= e^{\frac{x_1^2 + x_2^2}{2}}, & \mathcal{K} &= 0.1, \\ f &= \frac{(1 + x_1^2 + x_2^2)e^{x_1^2 + x_2^2}}{0.1(1 + (x_1^2 + x_2^2)e^{x_1^2 + x_2^2})^2}, & g &= e^{\frac{x_1^2 + x_2^2}{2}}. \\ \text{(b)} \quad u &= \cos(\sqrt{x_1}\pi) + \cos(\sqrt{x_2}\pi), & \mathcal{K} &= 0.025, \\ f &= \frac{\pi^2 \left(x_1^{-\frac{3}{2}} \sin(\sqrt{x_1}\pi) - x_1^{-1} \pi \cos(\sqrt{x_1}\pi) \right) \left(x_2^{-\frac{3}{2}} \sin(\sqrt{x_2}\pi) - x_2^{-1} \pi \cos(\sqrt{x_2}\pi) \right)}{16 \cdot 0.025 \left(1 + \frac{\pi^2}{4} (x_1^{-1} \sin^2(\sqrt{x_1}\pi) + x_2^{-1} \sin^2(\sqrt{x_2}\pi)) \right)^2}, \\ g &= \cos(\sqrt{x_1}\pi) + \cos(\sqrt{x_2}\pi). \end{aligned}$$

The computed solution, whose values are given in Table 4, is compared to the exact solution in Figure 11. As seen from Figure 11, the behavior of $\|u - u_h^\varepsilon\|$

² We note that it is easy to see the finite element methods and their convergence analyses of Section 6.2.1 and 6.2.2 also apply to the case $f > 0$ but $f \not\equiv 1$.

TABLE 4. Test 6.2.1: Error of $\|u - u_h^\varepsilon\|$ w.r.t. ε ($h = 0.01$) and estimated rate of convergence

	ε	$\ u - u_h^\varepsilon\ _{L^2}(\text{rate})$	$\ u - u_h^\varepsilon\ _{H^1}(\text{rate})$	$\ u - u_h^\varepsilon\ _{H^2}(\text{rate})$
Test 6.2.1a	1.0E-01	6.12E-02(—)	3.34E-01(—)	3.04E+00(—)
	5.0E-02	4.27E-02(0.52)	2.59E-01(0.37)	2.80E+00(0.12)
	2.5E-02	2.88E-02(0.57)	1.97E-01(0.39)	2.54E+00(0.14)
	1.0E-02	1.64E-02(0.62)	1.34E-01(0.42)	2.20E+00(0.16)
	5.0E-03	1.03E-02(0.66)	9.75E-02(0.46)	1.94E+00(0.18)
	2.5E-03	6.35E-03(0.70)	6.92E-02(0.49)	1.70E+00(0.19)
	1.0E-03	3.18E-03(0.75)	4.24E-02(0.53)	1.41E+00(0.21)
	5.0E-04	1.82E-03(0.80)	2.85E-02(0.58)	1.21E+00(0.22)
Test 6.2.1b	1.0E-01	2.84E-02(—)	1.95E-01(—)	2.51E+00(—)
	5.0E-02	1.87E-02(0.60)	1.47E-01(0.41)	2.27E+00(0.15)
	2.5E-02	1.20E-02(0.64)	1.08E-01(0.44)	2.02E+00(0.17)
	1.0E-02	6.34E-03(0.70)	6.92E-02(0.49)	1.70E+00(0.19)
	5.0E-03	3.78E-03(0.75)	4.80E-02(0.53)	1.47E+00(0.21)
	2.5E-03	2.19E-03(0.79)	3.24E-02(0.56)	1.27E+00(0.22)
	1.0E-03	1.02E-03(0.83)	1.87E-02(0.60)	1.03E+00(0.23)
	5.0E-04	5.56E-04(0.87)	1.20E-02(0.64)	8.74E-01(0.24)

behaves similarly to that of the Monge-Ampère equation, that is, we observe the following rates of convergence as $\varepsilon \rightarrow 0^+$:

$$\|u - u_h^\varepsilon\|_{L^2} \approx O(\varepsilon), \quad \|u - u_h^\varepsilon\|_{H^1} \approx O(\varepsilon^{\frac{3}{4}}), \quad \|u - u_h^\varepsilon\|_{H^2} \approx O(\varepsilon^{\frac{1}{4}}).$$

Since we have fixed h very small, we expect that $\|u - u^\varepsilon\|$ behaves similarly.

Test 6.2.2. In this test, we calculate u_h^ε using the Hermann-Miyoshi mixed finite element method developed in the previous subsection to calculate the rate of convergence of $\|u^\varepsilon - u_h^\varepsilon\|$ with respect to h for fixed ε . We also compare the numerical tests with Theorem 6.10. Since u^ε is generally not known, we solve the following problem (compare to (6.49)–(6.50)): find $(\sigma_h^\varepsilon, u_h^\varepsilon) \in W_{\phi^\varepsilon}^h \times Q_{g^\varepsilon}^h$ such that

$$(6.63) \quad (\sigma_h^\varepsilon, \kappa_h) + (\operatorname{div}(\kappa_h), \nabla u_h^\varepsilon) = G(\kappa_h) \quad \forall \kappa_h \in W_0^h,$$

$$(6.64) \quad (\operatorname{div}(\sigma_h^\varepsilon), \nabla z_h) + \left(\frac{\det(\sigma_h^\varepsilon)}{(1 + |\nabla u_h^\varepsilon|^2)^2}, z_h \right) = (\mathcal{K}f^\varepsilon, z_h) \quad \forall z_h \in Q_0^h,$$

where

$$\begin{aligned} W_{\phi^\varepsilon}^h &:= \{ \mu_h \in W^h; D^2 \mu_h \nu \cdot \nu|_{\partial\Omega} = \phi^\varepsilon \}, \\ Q_{g^\varepsilon}^h &:= \{ v_h \in Q^h; v_h|_{\partial\Omega} = g^\varepsilon \}. \end{aligned}$$

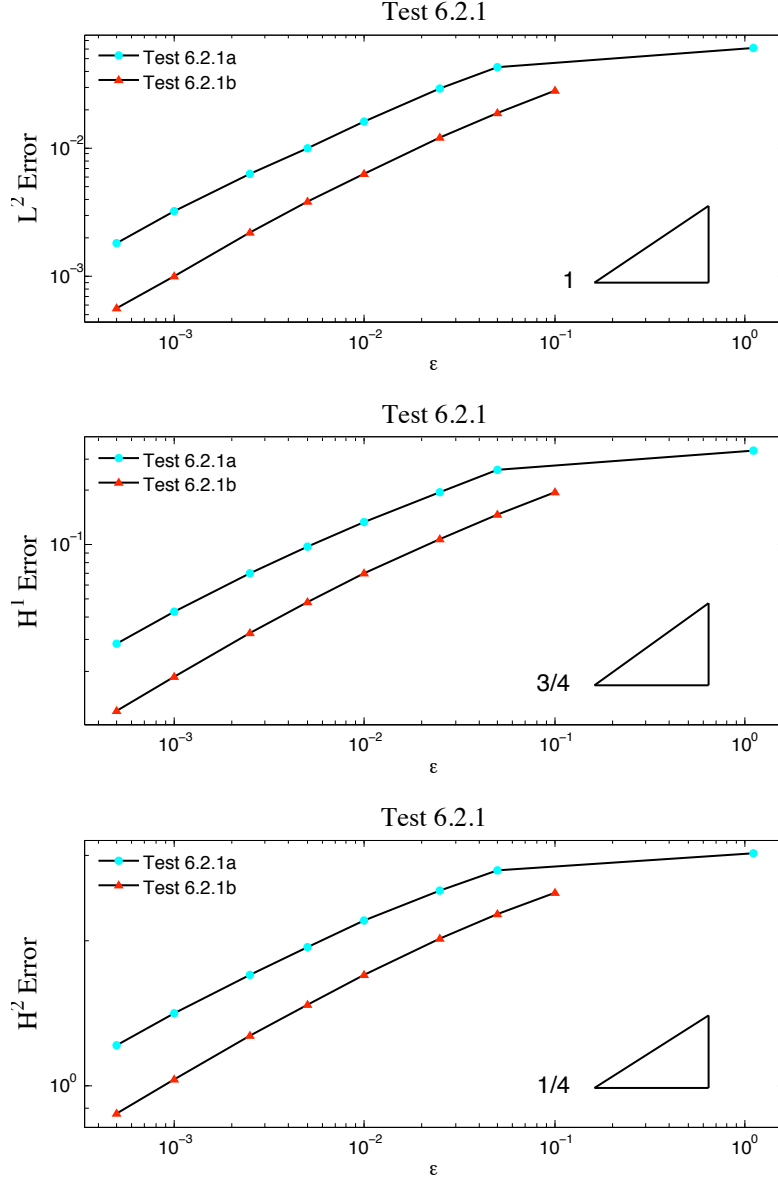


FIGURE 11. Test 6.2.1. Change of $\|u - u_h^\varepsilon\|$ w.r.t. ε ($h = 0.01$)

We use the following test functions and data:

$$\begin{aligned}
 \text{(a) } u^\varepsilon &= e^{\frac{x_1^2 + x_2^2}{2}}, & f^\varepsilon &= \frac{(1 + x_1^2 + x_2^2)e^{x_1^2 + x_2^2}}{0.1(1 + (x_1^2 + x_2^2)e^{x_1^2 + x_2^2})^2} \\
 & & & - \varepsilon(4(1 + x_1^2 + x_2^2) + (2 + x_1^2 + x_2^2)^2)e^{\frac{x_1^2 + x_2^2}{2}}, \\
 g^\varepsilon &= e^{\frac{x_1^2 + x_2^2}{2}}, & \phi^\varepsilon &= e^{\frac{x_1^2 + x_2^2}{2}} \left((1 + x_1^2)\nu_1^2 + 2x_1x_2\nu_1\nu_2 + (1 + x_2^2)\nu_2^2 \right) \\
 \mathcal{K} &= 0.1. \\
 \text{(b) } u^\varepsilon &= \frac{1}{8}(x_1^2 + x_2^2)^4, \quad \mathcal{K} = 0.1, & g^\varepsilon &= \frac{1}{8}(x_1^2 + x_2^2)^4,
 \end{aligned}$$

TABLE 5. Test 6.2.2: Error of $\|u^\varepsilon - u_h^\varepsilon\|$ w.r.t. h ($\varepsilon = 0.01$) and estimated rate of convergence

	h	$\ u^\varepsilon - u_h^\varepsilon\ _{L^2}(\text{rate})$	$\ u^\varepsilon - u_h^\varepsilon\ _{H^1}(\text{rate})$	$\ \sigma^\varepsilon - \sigma_h^\varepsilon\ _{L^2}(\text{rate})$
Test 6.2.2a	2.00E-01	2.04E-04(—)	5.98E-03(—)	4.40E-02(—)
	1.00E-01	2.60E-05(2.97)	1.52E-03(1.98)	1.68E-02(1.39)
	5.00E-02	3.28E-06(2.98)	3.72E-04(2.03)	6.07E-03(1.46)
	2.50E-02	4.16E-07(2.98)	9.25E-05(2.01)	2.19E-03(1.47)
	1.25E-02	5.24E-08(2.99)	2.31E-05(2.00)	7.87E-04(1.48)
Test 6.2.2b	2.00E-01	2.05E-03(—)	4.72E-02(—)	3.64E-01(—)
	1.00E-01	2.77E-04(2.89)	1.19E-02(1.99)	1.46E-01(1.32)
	5.00E-02	3.66E-05(2.92)	2.89E-03(2.04)	5.44E-02(1.42)
	2.50E-02	4.72E-06(2.95)	7.09E-04(2.03)	1.97E-02(1.47)
	1.25E-02	6.02E-07(2.97)	1.76E-04(2.01)	7.04E-03(1.48)

$$f^\varepsilon = \frac{7(6x_1^2x_2^2(x_1^8 + x_2^8) + 15x_1^4x_2^4(x_1^4 + x_2^4) + 20x_1^6x_2^6 + x_1^{12} + x_2^{12})}{0.1(1 + x_1^2(x_1^2 + x_2^2)^6 + x_2^2(x_2^2 + x_1^2)^6)^2} - 288\varepsilon(x_1^2 + x_2^2)^2,$$

$$\phi^\varepsilon = (7x_1^2 + x_2^2)(x_1^2 + x_2^2)^2\nu_1^2 + 12(x_1^2 + x_2^2)^2x_1x_2\nu_1\nu_2 + (7x_2^2 + x_1^2)(x_1^2 + x_2^2)^2\nu_2^2.$$

We record the results in Table 5 and plot the results in Figure 12. The data clearly indicates the following rates of convergence:

$$\|u^\varepsilon - u_h^\varepsilon\|_{H^1} = O(h^2), \quad \|u^\varepsilon - u_h^\varepsilon\|_{L^2} = O(h^3).$$

These are exactly theoretical rates of convergence proved at the beginning of this section, indicating that our theoretical estimates for $u^\varepsilon - u_h^\varepsilon$ are sharp. On the other hand, we note that the numerical rate is better than the theoretical estimate for $\sigma^\varepsilon - \sigma_h^\varepsilon$ which is expected because the theoretical rate of convergence for $\sigma^\varepsilon - \sigma_h^\varepsilon$ is clearly not optimal from the approximation point of view. This phenomenon also occurs when approximating the linear biharmonic equation by the Hermann-Miyoshi finite element method (cf. [35]).

Test 6.2.3. For this test, we use our numerical method to approximate \mathcal{K}^* and compare our results with those found in [4], where the method of continuity (which was used to prove the existence of classical solutions to the equation of prescribed Gauss curvature) was implemented at the discrete level. We compute (6.31)–(6.33) with the following Dirichlet boundary conditions and domains as used in [4]:

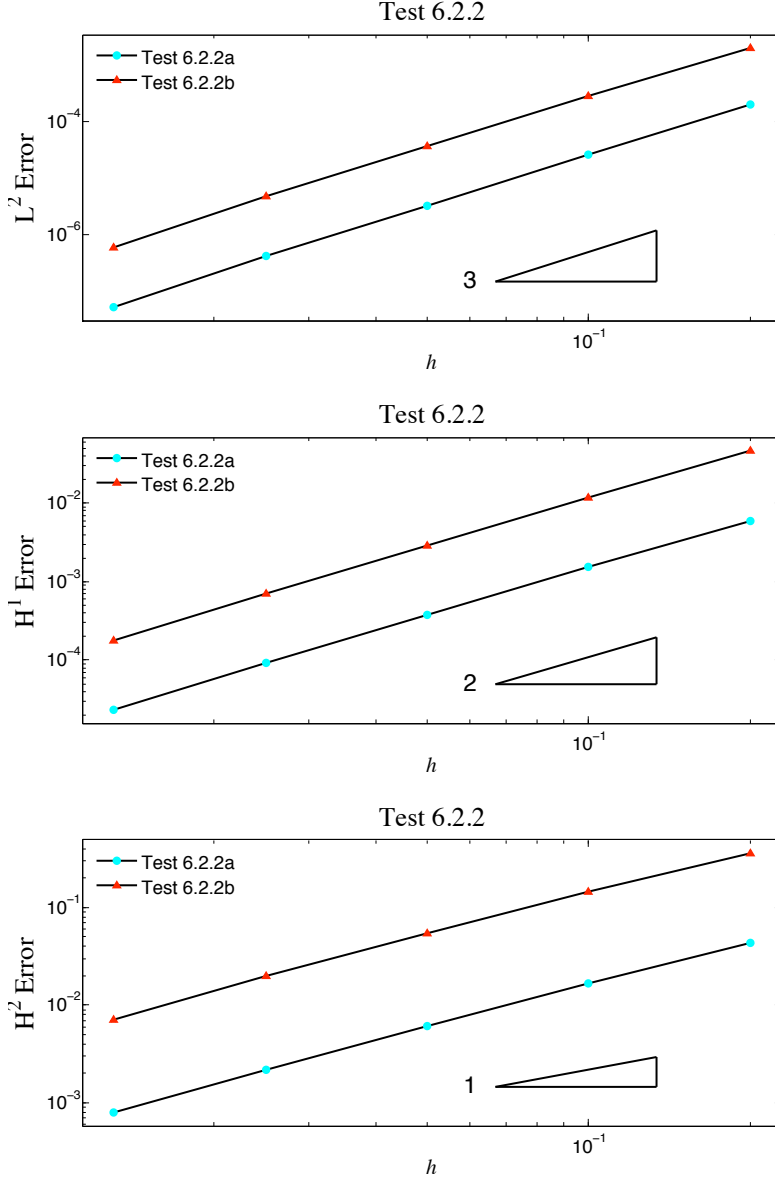


FIGURE 12. Test 6.2.2. Change of $\|u^\varepsilon - u_h^\varepsilon\|$ w.r.t. h ($\varepsilon = 0.01$)

- | | |
|---|---|
| (a) $g = \sqrt{1 - x_1^2 - x_2^2}$, | $\Omega = (-0.57, 0.57)^2$. |
| (b) $g = 1 - x_1^2 - x_2^2$, | $\Omega = (-0.57, 0.57)^2$. |
| (c) $g = 1 - (x_1 - 0.075)^2 - (x_2 - 0.015)^2$, | $\Omega = (-0.57, 0.57)^2$. |
| (d) $g = \sqrt{1 - x_1^2 - x_2^2}$, | $\Omega = (-0.72, 0.72) \times (-0.36, 0.36)$. |
| (e) $g = 1 - x_1^2 - x_2^2$, | $\Omega = (-0.72, 0.72) \times (-0.36, 0.36)$. |
| (f) $g = 1 - (x_1 - 0.075)^2 - (x_2 - 0.015)^2$, | $\Omega = (-0.72, 0.72) \times (-0.36, 0.36)$. |

TABLE 6. Test 6.2.3. Computed \mathcal{K}^* with $\varepsilon = -0.001$ and $h = 0.031$

	Computed \mathcal{K}^*	\mathcal{K}^* in [4]
Test 6.2.3a	2.07	2.10
Test 6.2.3b	2.20	2.24
Test 6.2.3c	1.95	1.85
Test 6.2.3d	2.68	2.61
Test 6.2.3e	2.71	2.73
Test 6.2.3f	2.20	2.27

We remark that for the above choice of data, the solution of the prescribed Gauss curvature equation is concave, and so we set $\varepsilon < 0$ in order to approximate the solution (see [37, 61] for further explanation). Table 6 compares our results and those of [4]. Table 6 shows that our numerical method gives comparable values to those computed in [4]. Finally, we plot the computed solution of Test 6.3a for \mathcal{K} -values 0, 1, and 2 in Figure 13. We also compute and plot the corresponding convex solution (with $g = -\sqrt{1 - x_1^2 - x_2^2}$ and $\varepsilon = 0.001$) for comparison.

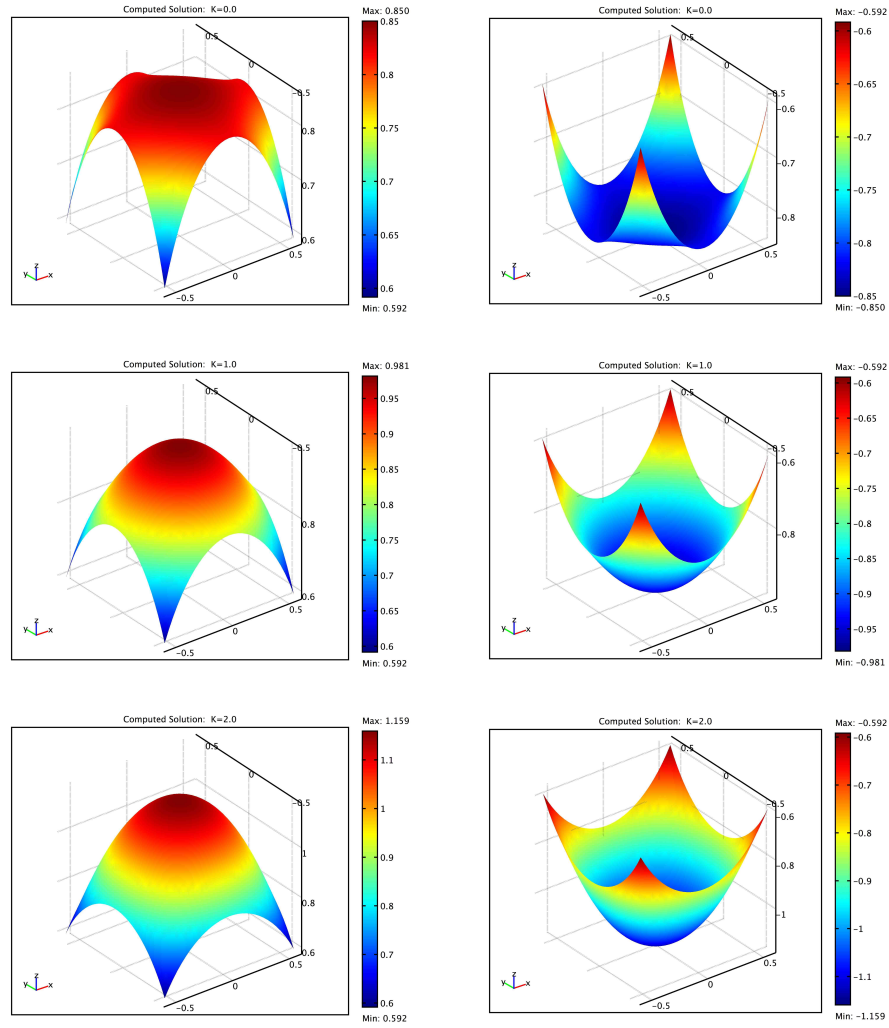


FIGURE 13. Test 6.2.3a. Computed concave solution (left) and convex solution (right) with $K = 0.0$ (top), $K = 1$ (middle), and $K = 2$ (bottom). $h = 0.025$ and $\varepsilon = -0.001$ to compute the concave solution, where as $h = 0.025$ and $\varepsilon = 0.001$ to compute the convex solution.

6.3. The infinity-Laplacian equation

In this section, we consider finite element approximations of the infinity-Laplacian equation:

$$(6.65) \quad \Delta_\infty u = 0 \quad \text{in } \Omega,$$

$$(6.66) \quad u = g \quad \text{on } \partial\Omega,$$

where

$$\Delta_\infty u := \frac{D^2 u \nabla u \cdot \nabla u}{|\nabla u|^2} = \frac{1}{|\nabla u|^2} \sum_{i,j=1}^n \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j},$$

and $g \in C(\partial\Omega)$. We note that unlike the PDEs considered in the previous two sections, the infinity-Laplacian equation is not fully nonlinear, but rather quasilinear. Still, its non-divergence form, degeneracy, and strong nonlinearity in the first order derivatives makes the PDE difficult to study and approximate ([8, 33, 63]). In particular, the linearization of the operator Δ_∞ gives a degenerate linear differential operator which serves as a perfect example for testing the mixed finite element theory developed in Section 5.4.

Remark 6.11. As pointed out in Section 6.1, both $\tilde{\Delta}_\infty v := D^2 v \nabla v \cdot \nabla v$ and $\Delta_\infty v := \frac{D^2 v \nabla v \cdot \nabla v}{|\nabla v|^2}$ are called the infinity-Laplacian in the literature [3, 34] because they give the same infinity-Laplacian equation. Here we adopt the latter definition for a reason which will be clear later (see Remark 6.14).

The infinity-Laplacian equation (6.65) arises from the so-called “absolute minimal problem” which is stated as follows: *Given a continuous function $g : \partial\Omega \mapsto \mathbf{R}$, find a function $u : \bar{\Omega} \mapsto \mathbf{R}$ such that for each $V \subset \Omega$ and each $v \in C(\bar{V})$ $u|_{\partial V} = v|_{\partial V}$ implies $\text{esssup}_V |\nabla u| \leq \text{esssup}_V |\nabla v|$.* The equation finds applications in image processing and many other fields, we refer the reader to two recent survey papers [3, 28] for detailed discussions on the latest developments on PDE analysis and applications of the infinity-Laplacian equation.

Like the equation of prescribed Gauss curvature, we have some flexibility in defining $F(D^2 u, \nabla u, u, x)$. One possibility is to define $F(D^2 u, \nabla u, u, x) := -\tilde{\Delta}_\infty u$, but this leads to difficulties in the linearization (see Remark 6.14). Here, we define

$$(6.67) \quad F(D^2 u, \nabla u, u, x) := -\frac{\tilde{\Delta}_\infty u}{|\nabla u|^2 + \gamma} = -\frac{D^2 u \nabla u \cdot \nabla u}{|\nabla u|^2 + \gamma},$$

where $\gamma > 0$ is a positive parameter that will be specified later. The reason for introducing γ is to avoid dividing by zero in the expression.

It is easy to check that

$$\begin{aligned} F'[v](w) &= -\frac{D^2 w \nabla v \cdot \nabla v + 2D^2 v \nabla v \cdot \nabla w}{|\nabla v|^2 + \gamma} + 2\frac{\tilde{\Delta}_\infty v \nabla v \cdot \nabla w}{(|\nabla v|^2 + \gamma)^2}, \\ F'[\mu, v](\kappa, w) &= -\frac{\kappa \nabla v \cdot \nabla v + 2\mu \nabla v \cdot \nabla w}{|\nabla v|^2 + \gamma} + 2\frac{(\mu \nabla v \cdot \nabla v) \nabla v \cdot \nabla w}{(|\nabla v|^2 + \gamma)^2}. \end{aligned}$$

The vanishing moment approximation becomes

$$(6.68) \quad \varepsilon \Delta^2 u^\varepsilon - \frac{\tilde{\Delta}_\infty u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma} = 0 \quad \text{in } \Omega,$$

$$(6.69) \quad u^\varepsilon = g \quad \text{on } \partial\Omega,$$

$$(6.70) \quad \Delta u^\varepsilon = \varepsilon \quad \text{on } \partial\Omega.$$

The linearization of

$$G_\varepsilon(u^\varepsilon) = \varepsilon \Delta^2 u^\varepsilon - \frac{\tilde{\Delta}_\infty u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma}$$

at the solution u^ε is

$$G'_\varepsilon[u^\varepsilon](v) = \varepsilon \Delta^2 v - \frac{D^2 v \nabla u^\varepsilon \cdot \nabla u^\varepsilon + 2D^2 u^\varepsilon \nabla u^\varepsilon \cdot \nabla v}{|\nabla u^\varepsilon|^2 + \gamma} + 2 \frac{\tilde{\Delta}_\infty u^\varepsilon \nabla u^\varepsilon \cdot \nabla v}{(|\nabla u^\varepsilon|^2 + \gamma)^2}.$$

Numerical tests indicate that there exists a unique solution to (6.68)–(6.70) (cf. Subsection 6.3.3 and [37]), and therefore, for the continuation of this section, we assume that there exists a unique solution to (6.68)–(6.70).

Before formulating and analyzing finite element methods for (6.68)–(6.70), we first state the following two identities.

Lemma 6.12. *Suppose that $n = 2$. Then there holds the following identity:*

$$|\nabla w|^2 (|\Delta w|^2 - |D^2 w|^2) = (\Delta w \nabla w - D^2 w \nabla w) \cdot \nabla (|\nabla w|^2).$$

The proof of of Lemma 6.12 is a straight-forward (and tedious) calculation, so we omit it. Next, with the help of Lemma 6.12, we are able to establish the following identity.

Lemma 6.13. *Suppose that $n = 2$. Then for any $v \in H_0^1(\Omega)$, there holds*

$$(6.71) \quad \langle F'[u^\varepsilon](v), v \rangle = \left\| \frac{\nabla v \cdot \nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2 + \gamma}} \right\|_{L^2}^2 - \gamma \left(\frac{\det(D^2 u^\varepsilon)}{(|\nabla u^\varepsilon|^2 + \gamma)^2}, v^2 \right).$$

PROOF. Integrating by parts we get

$$\begin{aligned} & \left(\frac{D^2 v \nabla u^\varepsilon \cdot \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma}, v \right) \\ &= - \left(\frac{\nabla v \cdot \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma}, \nabla v \cdot \nabla u^\varepsilon \right) - \left(\frac{D^2 u^\varepsilon \nabla u^\varepsilon \cdot \nabla v}{|\nabla u^\varepsilon|^2 + \gamma}, v \right) - \left(\frac{\nabla v \cdot \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma}, \Delta u^\varepsilon v \right) \\ & \quad + \left(\frac{\nabla v \cdot \nabla u^\varepsilon}{(|\nabla u^\varepsilon|^2 + \gamma)^2}, \nabla (|\nabla u^\varepsilon|^2) \cdot \nabla u^\varepsilon v \right) \\ &= - \left(\frac{\nabla v \cdot \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma}, \nabla v \cdot \nabla u^\varepsilon \right) - \left(\frac{D^2 u^\varepsilon \nabla u^\varepsilon \cdot \nabla v}{|\nabla u^\varepsilon|^2 + \gamma}, v \right) - \left(\frac{\nabla v \cdot \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma}, \Delta u^\varepsilon v \right) \\ & \quad + 2 \left(\frac{\nabla v \cdot \nabla u^\varepsilon}{(|\nabla u^\varepsilon|^2 + \gamma)^2}, \tilde{\Delta}_\infty u^\varepsilon v \right). \end{aligned}$$

Thus,

$$\begin{aligned}
& \langle F'[u^\varepsilon](v), v \rangle \\
&= - \left(\frac{D^2 v \nabla u^\varepsilon \cdot \nabla u^\varepsilon + 2D^2 u^\varepsilon \nabla u^\varepsilon \cdot \nabla v}{|\nabla u^\varepsilon|^2 + \gamma}, v \right) + 2 \left(\frac{\tilde{\Delta}_\infty u^\varepsilon \nabla u^\varepsilon \cdot \nabla v}{(|\nabla u^\varepsilon|^2 + \gamma)^2}, v \right) \\
&= \left(\frac{\nabla v \cdot \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma}, \nabla v \cdot \nabla u^\varepsilon \right) - \left(\frac{D^2 u^\varepsilon \nabla u^\varepsilon \cdot \nabla v}{|\nabla u^\varepsilon|^2 + \gamma}, v \right) + \left(\frac{\nabla v \cdot \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma}, \Delta u^\varepsilon v \right) \\
&= \left\| \frac{\nabla v \cdot \nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2 + \gamma}} \right\|_{L^2}^2 + \frac{1}{2} \left(\frac{\Delta u^\varepsilon \nabla u^\varepsilon - D^2 u^\varepsilon \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma}, \nabla(v^2) \right) \\
&= \left\| \frac{\nabla v \cdot \nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2 + \gamma}} \right\|_{L^2}^2 - \frac{1}{2} \left(\operatorname{div} \left(\frac{\Delta u^\varepsilon \nabla u^\varepsilon - D^2 u^\varepsilon \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma} \right), v^2 \right).
\end{aligned}$$

Noting that

$$\begin{aligned}
& \operatorname{div} \left(\frac{\Delta u^\varepsilon \nabla u^\varepsilon - D^2 u^\varepsilon \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma} \right) \\
&= \frac{|\Delta u^\varepsilon|^2 - |D^2 u^\varepsilon|^2}{|\nabla u^\varepsilon|^2 + \gamma} - \frac{(\Delta u^\varepsilon \nabla u^\varepsilon - D^2 u^\varepsilon \nabla u^\varepsilon) \cdot \nabla(|\nabla u^\varepsilon|^2)}{(|\nabla u^\varepsilon|^2 + \gamma)^2} \\
&= \frac{(|\nabla u^\varepsilon|^2 + \gamma)(|\Delta u^\varepsilon|^2 - |D^2 u^\varepsilon|^2) - (\Delta u^\varepsilon \nabla u^\varepsilon - D^2 u^\varepsilon \nabla u^\varepsilon) \cdot \nabla(|\nabla u^\varepsilon|^2)}{(|\nabla u^\varepsilon|^2 + \gamma)^2},
\end{aligned}$$

we have by Lemma 6.12,

$$\begin{aligned}
\langle F'[u^\varepsilon](v), v \rangle &= \left\| \frac{\nabla v \cdot \nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2 + \gamma}} \right\|_{L^2}^2 - \frac{\gamma}{2} \left(\frac{|\Delta u^\varepsilon|^2 - |D^2 u^\varepsilon|^2}{(|\nabla u^\varepsilon|^2 + \gamma)^2}, v^2 \right) \\
&= \left\| \frac{\nabla v \cdot \nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2 + \gamma}} \right\|_{L^2}^2 - \gamma \left(\frac{\det(D^2 u^\varepsilon)}{(|\nabla u^\varepsilon|^2 + \gamma)^2}, v^2 \right).
\end{aligned}$$

□

Remark 6.14. (a) Unlike the two PDEs analyzed in the previous sections, the operator $F'[u^\varepsilon]$ is not uniformly elliptic, that is, there does not exist constants $K_0, K_1 > 0$ such that

$$\langle F'[u^\varepsilon](v), v \rangle \geq K_1 \|v\|_{H^1} - K_0 \|v\|_{L^2}^2 \quad \forall v \in H_0^1(\Omega).$$

Thus, when constructing and analyzing mixed finite element methods for (6.68)–(6.70), we must instead use the abstract analysis of Section 5.4, which is developed exactly with such a case in mind.

(b) If we set $F(D^2 u, \nabla u, u, x) = -\tilde{\Delta}_\infty u := D^2 u \nabla u \cdot \nabla u$, then the linearization of F would be

$$F'[v](w) = -D^2 w \nabla v \cdot \nabla v - 2D^2 v \nabla v \cdot \nabla w,$$

and it is an easy exercise to see that

$$\langle F'[u^\varepsilon](v), v \rangle = \|\nabla v \cdot \nabla u^\varepsilon\|_{L^2}^2 - \frac{1}{2} (|\Delta u^\varepsilon|^2 - |D^2 u^\varepsilon|^2, v^2).$$

Thus, the reason we use the definition (6.67) is so that we are able to control the zeroth order term in the linearization as shown in the following corollary. Nevertheless, numerical experiments of [37, 61] indicate that the vanishing moment method with $F(D^2u, \nabla u, u, x) = -\tilde{\Delta}_\infty u$ also work well for the infinity-Laplacian equation.

Corollary 6.15. *Suppose $n = 2$. Then there exists a constant $\gamma_0 = \gamma_0(\varepsilon) > 0$, such that for $\gamma \in (0, \gamma_0]$, there holds*

$$(6.72) \quad \langle G'_\varepsilon[u^\varepsilon](v), v \rangle \geq C\varepsilon \|v\|_{H^2}^2 \quad \forall v \in V_0.$$

PROOF. If $\|\nabla u^\varepsilon\|_{L^\infty} \neq 0$, then by (6.71), we have

$$\begin{aligned} \langle G'_\varepsilon[u^\varepsilon](v), v \rangle &\geq C\varepsilon \|v\|_{H^2}^2 + \left\| \frac{\nabla v \cdot \nabla u^\varepsilon}{\sqrt{|\nabla u^\varepsilon|^2 + \gamma}} \right\|_{L^2}^2 - \gamma \left(\frac{\det(D^2 u^\varepsilon)}{(|\nabla u^\varepsilon|^2 + \gamma)^2}, v^2 \right) \\ &\geq C\varepsilon \|v\|_{H^2}^2 - \gamma \left\| \frac{\det(D^2 u^\varepsilon)}{(|\nabla u^\varepsilon|^2 + \gamma)^2} \right\|_{L^\infty} \|v\|_{L^2}^2 \\ &\geq C \left(\varepsilon - \gamma \frac{\|u^\varepsilon\|_{W^{2,\infty}}^2}{\|\nabla u^\varepsilon\|_{L^\infty}^4} \right) \|v\|_{H^2}^2. \end{aligned}$$

Choosing $\gamma_0 = \frac{\varepsilon \|\nabla u^\varepsilon\|_{L^\infty}^4}{2\|u^\varepsilon\|_{W^{2,\infty}}^2}$, we get the desired result.

On the other hand, if $\|\nabla u^\varepsilon\|_{L^\infty} = 0$, then $u^\varepsilon \equiv \text{const}$ and $F'[u^\varepsilon] \equiv 0$, then we can choose γ_0 to be any positive number to obtain

$$\langle G'_\varepsilon[u^\varepsilon](v), v \rangle = \varepsilon \|\Delta v\|_{L^2}^2.$$

□

6.3.1. Conforming finite element methods for the infinity-Laplacian equation. The finite element method for (6.68)–(6.70) is defined as finding $u_h^\varepsilon \in V_g^h$ such that

$$(6.73) \quad \varepsilon(\Delta u_h^\varepsilon, \Delta v_h) - \left(\frac{\tilde{\Delta}_\infty u_h^\varepsilon}{|\nabla u_h^\varepsilon|^2 + \gamma}, v_h \right) = \left\langle \varepsilon^2, \frac{\partial v_h}{\partial \nu} \right\rangle_{\partial \Omega} \quad \forall v_h \in V_0^h,$$

where we assume that $\gamma \in (0, \gamma_0]$ for the rest of this subsection so that the inequality (6.72) holds. Furthermore, we assume that $\|\nabla u^\varepsilon\|_{L^\infty} \geq 1$. This assumption is not necessary in our analysis, but it does simplify our presentation (cf. (6.76)).

The goal of this section is to apply the abstract framework of Chapter 4 to the finite element method (6.73). Specifically, we now show that conditions [A1]–[A5] hold, which will then gives us the existence, uniqueness, and error estimates of the solution to (6.73). Of particular interest is the constants' explicit dependence on ε in the error estimates. We summarize our findings in the following theorem.

Theorem 6.16. *Suppose $n = 2$, and let $u^\varepsilon \in H^s(\Omega)$ be the solution to (6.68)–(6.70) with $s \geq 3$. Then there exists an $h_3 = h_3(\varepsilon) > 0$ such that for $h \leq h_3$, (6.73) has a unique solution. Furthermore, there holds the following error estimates:*

$$(6.74) \quad \|u^\varepsilon - u_h^\varepsilon\|_{H^2} \leq C_7 h^{\ell-2} \|u^\varepsilon\|_{H^\ell},$$

$$(6.75) \quad \|u^\varepsilon - u_h^\varepsilon\|_{L^2} \leq C_8 \left(C_2 h^\ell \|u^\varepsilon\|_{H^\ell} + C_7 L(h) h^{2\ell-4} \|u^\varepsilon\|_{H^\ell} \right),$$

where

$$C_2 = |u^\varepsilon|_{W^{2, \frac{q}{q-1}}}, \quad C_7 = C\varepsilon^{-2} \gamma^{-\frac{1}{2}} |u^\varepsilon|_{W^{2,\infty}} |u^\varepsilon|_{W^{2, \frac{q}{q-1}}}, \quad C_8 = CC_7 C_R.$$

where q is a number in the interval $(1, \infty)$, C_R is defined by (6.77), $L(h)$ is defined by (6.87), $\ell = \min\{s, k+1\}$, and k denotes the polynomial degree of the finite element space.

PROOF. First, Corollary 6.15 implies that $(G'_\varepsilon[u^\varepsilon])^*$ is an isomorphism from V_0 to V_0^* .

Next, we note that

$$(6.76) \quad \left\| \frac{|\nabla u^\varepsilon|^p}{(|\nabla u^\varepsilon|^2 + \gamma)^m} \right\|_{L^\infty} \leq \|\nabla u^\varepsilon\|_{L^\infty}^{p-2m} \leq 1, \quad 2m \geq p \geq 1.$$

Thus, for any $v, w \in V_0$, we have by using Sobolev inequalities for any $q \in (1, \infty)$

$$\begin{aligned} \langle F'[u^\varepsilon](v), w \rangle &= \left(\frac{\nabla u^\varepsilon \cdot \nabla v}{|\nabla u^\varepsilon|^2 + \gamma}, \nabla u^\varepsilon \cdot \nabla w \right) - \left(\frac{D^2 u^\varepsilon \nabla u^\varepsilon \cdot \nabla v - \nabla v \cdot \nabla u^\varepsilon \Delta u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma}, w \right) \\ &\leq \left\| \frac{|\nabla u^\varepsilon|^2}{|\nabla u^\varepsilon|^2 + \gamma} \right\|_{L^\infty} \|\nabla v\|_{L^2} \|\nabla w\|_{L^2} + \left\| \frac{D^2 u^\varepsilon \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma} \right\|_{L^{\frac{q}{q-1}}} \|\nabla v\|_{L^q} \|w\|_{L^\infty} \\ &\quad + \left\| \frac{|\nabla u^\varepsilon| \Delta u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma} \right\|_{L^{\frac{q}{q-1}}} \|\nabla v\|_{L^q} \|w\|_{L^\infty} \\ &\leq C \left(\|\nabla v\|_{L^2} \|\nabla w\|_{L^2} + \|D^2 u^\varepsilon\|_{L^{\frac{q}{q-1}}} \|\nabla v\|_{L^q} \|w\|_{L^\infty} \right) \\ &\leq C \|D^2 u^\varepsilon\|_{L^{\frac{q}{q-1}}} \|v\|_{H^2} \|w\|_{H^2}. \end{aligned}$$

Next, by the standard PDE theory, if we assume that u^ε and $\partial\Omega$ are sufficiently smooth, and if $v \in V_0$ solves

$$\langle G'_\varepsilon[u^\varepsilon](v), w \rangle = (\varphi, w) \quad \forall w \in V_0,$$

where φ is some $L^2(\Omega)$ function, then $v \in H^p(\Omega)$ for $p \geq 3$. Furthermore, in view of Remark 4.4, and the inequalities (which come from (6.76))

$$\begin{aligned} \left\| \frac{\partial F(u^\varepsilon)}{\partial r_{ij}} \right\|_{L^\infty} &= \left\| \frac{\frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial u^\varepsilon}{\partial x_j}}{|\nabla u^\varepsilon|^2 + \gamma} \right\|_{L^\infty} \leq \left\| \frac{|\nabla u^\varepsilon|^2}{|\nabla u^\varepsilon|^2 + \gamma} \right\|_{L^\infty} \leq C, \\ \left\| \frac{\partial F(u^\varepsilon)}{\partial p_i} \right\|_{L^\infty} &\leq 2 \left\| \frac{(D^2 u^\varepsilon \nabla u^\varepsilon)_i}{|\nabla u^\varepsilon|^2 + \gamma} \right\|_{L^\infty} + 2 \left\| \frac{\Delta u^\varepsilon (D^2 u^\varepsilon \nabla u^\varepsilon)_i}{(|\nabla u^\varepsilon|^2 + \gamma)^2} \right\|_{L^\infty} \\ &\leq C \left(\|D^2 u^\varepsilon\|_{L^\infty} \left\| \frac{|\nabla u^\varepsilon|}{|\nabla u^\varepsilon|^2 + \gamma} \right\|_{L^\infty} + \|D^2 u^\varepsilon\|_{L^\infty}^2 \left\| \frac{|\nabla u^\varepsilon|^3}{(|\nabla u^\varepsilon|^2 + \gamma)^2} \right\|_{L^\infty} \right) \\ &\leq C \|D^2 u^\varepsilon\|_{L^\infty}^2, \end{aligned}$$

we have that in the case $p = 4$

$$\|v\|_{H^4} \leq C\varepsilon^{-2} \|D^2 u^\varepsilon\|_{L^\infty}^2 \|\varphi\|_{L^2}.$$

It then follows that condition [A2] holds with

$$(6.77) \quad \begin{aligned} C_0 &= C\varepsilon, & C_1 &= C\varepsilon, & C_2 &= C \|D^2 u^\varepsilon\|_{L^{\frac{q}{q-1}}}, \\ p &= 4, & C_R &= C\varepsilon^{-2} \|u^\varepsilon\|_{W^{2,\infty}}^2. \end{aligned}$$

It then follows from Theorem 4.3 that

$$(6.78) \quad C_4 = CC_2\varepsilon^{-1}, \quad C_5 = CC_2^2 C_R \varepsilon^{-1}, \quad h_0 = C(C_2 C_R)^{-\frac{1}{2}}.$$

To confirm [A3]–[A4], we set

$$(6.79) \quad Y = W^{2, \frac{q}{q-1}}(\Omega), \quad \|\cdot\|_Y = \gamma^{-\frac{1}{2}} \|\cdot\|_{W^{2, \frac{q-1}{q}}},$$

where $q \in (1, \infty)$.

Using a Sobolev inequality and the inequality (6.76), we have for any $y \in Y$, $v, w \in V_0$

$$\begin{aligned} \langle F'[y](v), w \rangle &= \left(\frac{\nabla y \cdot \nabla v}{|\nabla y|^2 + \gamma}, \nabla y \cdot \nabla w \right) - \left(\frac{D^2 y \nabla y \cdot \nabla v - \nabla v \cdot \nabla y \Delta y}{|\nabla y|^2 + \gamma}, w \right) \\ &\leq C \left(\left\| \frac{|\nabla y|^2}{|\nabla y|^2 + \gamma} \right\|_{L^\infty} \|\nabla v\|_{L^2} \|\nabla w\|_{L^2} \right. \\ &\quad \left. + \left\| \frac{|\nabla y|}{|\nabla y|^2 + \gamma} \right\|_{L^\infty} \|D^2 y\|_{L^{\frac{q}{q-1}}} \|\nabla v\|_{L^q} \|w\|_{L^\infty} \right) \\ &\leq C \gamma^{-\frac{1}{2}} \|D^2 y\|_{L^{\frac{q}{q-1}}} \|v\|_{H^2} \|w\|_{H^2}. \end{aligned}$$

Here, we have used the fact that for $p \leq 2m$ and $x \geq 0$, $\left| \frac{x^p}{(x^2 + \gamma)^m} \right| \leq C \gamma^{\frac{p-2m}{2}}$ for some constant that only depends on p and m .

It then follows from this calculation that

$$\sup_{y \in Y} \frac{\|F'[y]\|_{V^*}}{\|y\|_Y} \leq C.$$

Thus, [A3]–[A4] holds.

To verify condition [A5], we first note for any $v_h \in V_g^h$ and $w \in V_0$

(6.80)

$$\begin{aligned} &(F'[u^\varepsilon] - F'[v_h])(w) \\ &= \left(\frac{D^2 v_h \nabla v_h}{|\nabla v_h|^2 + \gamma} - \frac{D^2 u^\varepsilon \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma} \right) \cdot \nabla w + \frac{D^2 w \nabla v_h \cdot \nabla v_h}{|\nabla v_h|^2 + \gamma} - \frac{D^2 w \nabla u^\varepsilon \cdot \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma} \\ &\quad + \frac{2\tilde{\Delta}_\infty u^\varepsilon \nabla u^\varepsilon \cdot \nabla w}{(|\nabla u^\varepsilon|^2 + \gamma)^2} - \frac{2\tilde{\Delta}_\infty v_h \nabla v_h \cdot \nabla w}{(|\nabla v_h|^2 + \gamma)^2} \\ &= \frac{(D^2 v_h \nabla v_h - D^2 u^\varepsilon \nabla u^\varepsilon) \cdot \nabla w}{|\nabla v_h|^2 + \gamma} - D^2 u^\varepsilon \nabla u^\varepsilon \cdot \nabla w \left(\frac{|\nabla v_h|^2 - |\nabla u^\varepsilon|^2}{(|\nabla u^\varepsilon|^2 + \gamma)(|\nabla v_h|^2 + \gamma)} \right) \\ &\quad + \frac{D^2 w \nabla v_h \cdot \nabla v_h - D^2 w \nabla u^\varepsilon \cdot \nabla u^\varepsilon}{|\nabla v_h|^2 + \gamma} - D^2 w \nabla u^\varepsilon \cdot \nabla u^\varepsilon \left(\frac{|\nabla v_h|^2 - |\nabla u^\varepsilon|^2}{(|\nabla u^\varepsilon|^2 + \gamma)(|\nabla v_h|^2 + \gamma)} \right) \\ &\quad + \frac{2(\tilde{\Delta}_\infty u^\varepsilon \nabla u^\varepsilon - \tilde{\Delta}_\infty v_h \nabla v_h) \cdot \nabla w}{(|\nabla v_h|^2 + \gamma)^2} + 2\tilde{\Delta}_\infty u^\varepsilon \nabla u^\varepsilon \cdot \nabla w \left(\frac{1}{(|\nabla u^\varepsilon|^2 + \gamma)^2} - \frac{1}{(|\nabla v_h|^2 + \gamma)^2} \right). \end{aligned}$$

Bounding the second, fourth, and sixth term on the right-hand side of (6.80), we use Sobolev inequalities to conclude that for $\|\mathcal{I}^h u^\varepsilon - v_h\|_{H^2} \leq \delta \in (0, \frac{1}{2})$ and for any $q \in (1, \infty)$

$$\begin{aligned} (6.81) \quad &\left\| D^2 u^\varepsilon \nabla u^\varepsilon \cdot \nabla w \left(\frac{|\nabla v_h|^2 - |\nabla u^\varepsilon|^2}{(|\nabla u^\varepsilon|^2 + \gamma)(|\nabla v_h|^2 + \gamma)} \right) \right\|_{L^1} \\ &= \left\| D^2 u^\varepsilon \nabla u^\varepsilon \cdot \nabla w \left(\frac{(\nabla v_h - \nabla u^\varepsilon) \cdot (\nabla v_h + \nabla u^\varepsilon)}{(|\nabla u^\varepsilon|^2 + \gamma)(|\nabla v_h|^2 + \gamma)} \right) \right\|_{L^1} \\ &\leq C \|D^2 u^\varepsilon\|_{L^{\frac{q}{q-1}}} \|u^\varepsilon\|_{H^2}^2 \|u^\varepsilon - v_h\|_{H^2} \|w\|_{H^2}. \end{aligned}$$

Here, we have used that fact that if $\|\mathcal{I}^h u^\varepsilon - v_h\|_{H^2} \leq \delta \in (0, \frac{1}{2})$ and $\|\nabla u^\varepsilon\|_{L^\infty} \geq 1$, then $\|\nabla v_h\|_{L^\infty} \geq C$ for some positive constant C that is independent of h, ε , and γ .

Similarly,

(6.82)

$$\left\| D^2 w \nabla u^\varepsilon \cdot \nabla u^\varepsilon \left(\frac{|\nabla v_h|^2 - |\nabla u^\varepsilon|^2}{(|\nabla u^\varepsilon|^2 + \gamma)(|\nabla v_h|^2 + \gamma)} \right) \right\|_{L^1} \leq C \|u^\varepsilon\|_{H^2}^3 \|u^\varepsilon - v_h\|_{H^2} \|w\|_{H^2},$$

and

(6.83)

$$\begin{aligned} & \left\| \tilde{\Delta}_\infty u^\varepsilon \nabla u^\varepsilon \cdot \nabla w \left(\frac{1}{(|\nabla u^\varepsilon|^2 + \gamma)^2} - \frac{1}{(|\nabla v_h|^2 + \gamma)^2} \right) \right\|_{L^1} \\ &= \left\| \tilde{\Delta}_\infty u^\varepsilon \nabla u^\varepsilon \cdot \nabla w \left(\frac{(|\nabla v_h|^2 + |\nabla u^\varepsilon|^2 + 2\gamma)(\nabla u^\varepsilon - \nabla v_h)(\nabla u^\varepsilon + \nabla v_h)}{(|\nabla u^\varepsilon|^2 + \gamma)^2(|\nabla v_h|^2 + \gamma)^2} \right) \right\|_{L^1} \\ &\leq C \|D^2 u^\varepsilon\|_{L^{\frac{q}{q-1}}} \|u^\varepsilon\|_{H^2}^6 \|u^\varepsilon - v_h\|_{H^2} \|w\|_{H^2}. \end{aligned}$$

Bounding the first term in (6.80), we use similar techniques to conclude

$$\begin{aligned} (6.84) \quad \left\| \frac{(D^2 v_h \nabla v_h - D^2 u^\varepsilon \nabla u^\varepsilon) \cdot \nabla w}{|\nabla v_h|^2 + \gamma} \right\|_{L^1} &\leq C \gamma^{-1} \left(\|D^2 v_h - D^2 u^\varepsilon\|_{L^1} \|\nabla u^\varepsilon \cdot \nabla w\|_{L^1} \right. \\ &\quad \left. + \|D^2 v_h (\nabla u^\varepsilon - \nabla v_h) \cdot \nabla w\|_{L^1} \right) \\ &\leq C \|u^\varepsilon\|_{H^2} \|u^\varepsilon - v_h\|_{H^2} \|w\|_{H^2}. \end{aligned}$$

To bound the third term in (6.80), we use the identity

$$D^2 w \nabla v_h \cdot \nabla v_h - D^2 w \nabla u^\varepsilon \cdot \nabla u^\varepsilon = D^2 w (\nabla v_h + \nabla u^\varepsilon) \cdot (\nabla v_h - \nabla u^\varepsilon),$$

to obtain

$$(6.85) \quad \left\| \frac{D^2 w \nabla v_h \cdot \nabla v_h - D^2 w \nabla u^\varepsilon \cdot \nabla u^\varepsilon}{|\nabla v_h|^2 + \gamma} \right\|_{L^1} \leq C \|u^\varepsilon\|_{H^2} \|u^\varepsilon - v_h\|_{H^2} \|w\|_{H^2}.$$

Next, we write

$$\begin{aligned} & (\tilde{\Delta}_\infty u^\varepsilon \nabla u^\varepsilon - \tilde{\Delta}_\infty v_h \nabla v_h) \cdot \nabla w \\ &= (\tilde{\Delta}_\infty u^\varepsilon - \tilde{\Delta}_\infty v_h) \nabla v_h \cdot \nabla w + \tilde{\Delta}_\infty u^\varepsilon (\nabla u^\varepsilon - \nabla v_h) \cdot \nabla w \\ &= \left((D^2 u^\varepsilon - D^2 v_h) \nabla v_h \cdot \nabla v_h + D^2 u^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon - D^2 u^\varepsilon \nabla v_h \cdot \nabla v_h \right) \nabla v_h \cdot \nabla w \\ &\quad + \tilde{\Delta}_\infty u^\varepsilon (\nabla u^\varepsilon - \nabla v_h) \cdot \nabla w \\ &= \left((D^2 u^\varepsilon - D^2 v_h) \nabla v_h \cdot \nabla v_h + D^2 u^\varepsilon (\nabla u^\varepsilon + \nabla v_h) \cdot (\nabla u^\varepsilon - \nabla v_h) \right) \nabla v_h \cdot \nabla w \\ &\quad + \tilde{\Delta}_\infty u^\varepsilon (\nabla u^\varepsilon - \nabla v_h) \cdot \nabla w, \end{aligned}$$

so that

(6.86)

$$\left\| \frac{\tilde{\Delta}_\infty u^\varepsilon \nabla u^\varepsilon - \tilde{\Delta}_\infty v_h \nabla v_h}{(|\nabla v_h|^2 + \gamma)^2} \cdot \nabla w \right\|_{L^1} \leq C \|D^2 u^\varepsilon\|_{L^{\frac{q}{q-1}}} \|u^\varepsilon\|_{H^2}^2 \|u^\varepsilon - v_h\|_{H^2} \|w\|_{H^2}.$$

Applying the bounds (6.81)–(6.86) to the identity (6.80), we obtain

$$\begin{aligned} \|F'[u^\varepsilon] - F'[v_h]\|_{VV^*} &= \sup_{w \in V_0} \sup_{z \in V_0} \frac{\langle (F'[u^\varepsilon] - F'[v_h])(w), z \rangle}{\|w\|_{H^2} \|z\|_{H^2}} \\ &\leq \sup_{w \in V_0} \sup_{z \in V_0} \frac{\|F'[u^\varepsilon] - F'[v_h]\|_{L^1} \|z\|_{L^\infty}}{\|w\|_{H^2} \|z\|_{H^2}} \\ &\leq C \|D^2 u^\varepsilon\|_{L^{\frac{q}{q-1}}} \|u^\varepsilon\|_{H^2}^6 \|u^\varepsilon - v_h\|_{H^2}. \end{aligned}$$

Hence [A5] holds with

$$(6.87) \quad L(h) = C \|D^2 u^\varepsilon\|_{L^{\frac{q}{q-1}}} \|u^\varepsilon\|_{H^2}^6 \|u^\varepsilon - v_h\|_{H^2}$$

for any $q \in (1, \infty)$.

Gathering all of our results, existence and uniqueness of a solution to the finite element method (6.73) and the error estimates (6.74)–(6.75) follow from Theorem 4.7 and the estimates (6.77)–(6.79). \square

6.3.2. Mixed finite element methods for the infinity-Laplacian equation. As noted in the previous subsection, $F'[u^\varepsilon]$ is possibly degenerate, and therefore we need to resort to the abstract formulation and analysis of Section 5.4 for mixed finite element approximations of the infinity-Laplacian equation.

The mixed finite element method for (6.68)–(6.70) is then defined as follows: find $(\tilde{\sigma}_h^\varepsilon, u_h^\varepsilon) \in \widetilde{W}_\varepsilon^h \times Q_h^h$ such that

$$(6.88) \quad (\tilde{\sigma}_h^\varepsilon, \mu_h) + \tilde{b}(\mu_h, u_h^\varepsilon) = G(\mu_h) \quad \forall \mu_h \in W_0^h,$$

$$(6.89) \quad \tilde{b}(\tilde{\sigma}_h^\varepsilon, v_h) - \varepsilon^{-1} \tilde{c}(\tilde{\sigma}_h^\varepsilon, u_h^\varepsilon, v_h) = 0 \quad \forall v_h \in Q_0^h,$$

where $\tau \in (0, \tau_0)$ (τ_0 is defined in Lemma 5.12)

$$\begin{aligned} \tilde{b}(\kappa_h, u_h^\varepsilon) &= (\operatorname{div}(\kappa_h), \nabla u_h^\varepsilon), \\ \tilde{c}(\sigma_h^\varepsilon, u_h^\varepsilon, z_h) &= (\tilde{F}(\sigma_h^\varepsilon, u_h^\varepsilon), z_h), \\ \tilde{F}(\tilde{\sigma}_h^\varepsilon, u_h^\varepsilon) &= -2\varepsilon\tau\Delta u^\varepsilon - \varepsilon n\tau^2 u^\varepsilon - \frac{\tilde{\sigma}_h^\varepsilon \nabla u_h^\varepsilon \cdot \nabla u_h^\varepsilon}{|\nabla u_h^\varepsilon|^2 + \gamma} + \tau \frac{u_h^\varepsilon |\nabla u_h^\varepsilon|^2}{|\nabla u_h^\varepsilon|^2 + \gamma}. \end{aligned}$$

We also recall that

$$\widetilde{W}_\varepsilon^h = \{\mu_h \in W^h; \mu_h \nu \cdot \nu|_{\partial\Omega} = \varepsilon + \tau g\}.$$

The goal of this section is to apply the abstract analysis of Section 5.4 to the mixed method (6.88)–(6.89). We summarize our findings in the following theorem.

Theorem 6.17. *Let $u^\varepsilon \in H^s(\Omega)$ be the solution to (6.68)–(6.70) and let $\tilde{\sigma}^\varepsilon = D^2 u^\varepsilon + \tau I_{n \times n} u^\varepsilon$ with $\tau \in (0, \tau_0)$, where τ_0 is defined in Lemma 5.12. Then there exists $h_4 = h_4(\varepsilon) > 0$ such that for $h \leq h_4$ there exists a unique solution to (6.88)–(6.89). Furthermore, there holds the following error estimates:*

$$(6.90) \quad \|(\tilde{\sigma}^\varepsilon - \tilde{\sigma}_h^\varepsilon, u^\varepsilon - u_h^\varepsilon)\|_{\tilde{\varepsilon}} \leq \tilde{K}_8 h^{\ell-2} \|u^\varepsilon\|_{H^\ell},$$

$$(6.91) \quad \|u^\varepsilon - u_h^\varepsilon\|_{H^1} \leq K_{R_1} \left(\tilde{K}_9 h^{\ell-1} \|u^\varepsilon\|_{H^\ell} + \tilde{K}_8^2 R(h) h^{2\ell-4} \|u^\varepsilon\|_{H^\ell}^2 \right),$$

where

$$\begin{aligned} \|(\mu, v)\|_{\tilde{\varepsilon}} &= h\|\mu\|_{H^1} + \|\mu\|_{L^2} + \tau^{\frac{1}{2}}\|v\|_{H^1}, \\ \tilde{K}_8 &= C\tilde{K}_3\varepsilon^{-\frac{1}{2}}\left(\tau^{-\frac{1}{2}} + \varepsilon^{-\frac{3}{2}}|u^\varepsilon|_{W^{2,\infty}}^2\right), \quad \tilde{K}_9 = C\tilde{K}_8K_G, \\ \ell &= \min\{s, k+1\}. \end{aligned}$$

\tilde{K}_3 is defined by (6.93), K_{R_1} is defined by (6.92), and K_G is defined by (6.99).

PROOF. First, by (6.76) for any $v, z \in Q_0$ and for any $q \in (2, \infty)$

$$\begin{aligned} \langle F'[\sigma^\varepsilon, u^\varepsilon](D^2v, v), z \rangle &= \left(\frac{\nabla u^\varepsilon \cdot \nabla v}{(|\nabla u^\varepsilon|^2 + \gamma)}, \nabla u^\varepsilon \cdot \nabla w \right) \\ &\quad - \left(\frac{\sigma^\varepsilon \nabla u^\varepsilon \cdot \nabla v - \nabla v \cdot \nabla u^\varepsilon \operatorname{tr}(\sigma^\varepsilon)}{(|\nabla u^\varepsilon|^2 + \gamma)}, w \right) \\ &\leq \left\| \frac{|\nabla u^\varepsilon|^2}{|\nabla u^\varepsilon|^2 + \gamma} \right\|_{L^\infty} \|\nabla v\|_{L^2} \|\nabla w\|_{L^2} \\ &\quad + \left\| \frac{|\nabla u^\varepsilon|}{|\nabla u^\varepsilon|^2 + \gamma} \right\|_{L^\infty} \|\nabla v\|_{L^2} \|\sigma^\varepsilon\|_{L^{\frac{2q}{q-2}}} \|w\|_{L^q} \\ &\leq C\|\sigma^\varepsilon\|_{L^{\frac{2q}{q-2}}} \|v\|_{H^1} \|w\|_{H^1}. \end{aligned}$$

From this calculation, we conclude

$$\|F'[\sigma^\varepsilon, u^\varepsilon]\|_{QQ^*} \leq C\|\sigma^\varepsilon\|_{L^{\frac{2q}{q-2}}}$$

for some $q \in (2, \infty)$.

Therefore, using the same arguments as those used in the proof of Theorem 6.16, we can conclude that condition [B2] holds with

$$\begin{aligned} (6.92) \quad K_0 &= C\varepsilon, & K_2 &= C\|\sigma^\varepsilon\|_{L^{\frac{2q}{q-2}}}, \\ K_{R_0} &= C\varepsilon^{-2}|u^\varepsilon|_{W^{2,\infty}}^2, & p &= 4, \\ K_{R_1} &= C\varepsilon^{-2}. \end{aligned}$$

Next, to confirm [B3]–[B4], we set

$$X = \left[L^{\frac{2q}{q-2}}(\Omega) \right]^{n \times n}, \quad Y = W^{1,1}(\Omega),$$

$$\|(\omega, y)\|_{X \times Y} = \gamma^{-\frac{1}{2}} \|\omega\|_{L^{\frac{2q}{q-2}}} \quad \forall \omega \in X, y \in Y.$$

where q is any number in the interval $(2, \infty)$. We then have for any $\omega \in X$, $y \in Y$, $\chi \in W$, $v \in Q$, $z \in Q_0$,

$$\begin{aligned} &\langle F'[\omega, y](\chi, v), z \rangle \\ &= - \left(\frac{\chi \nabla y \cdot \nabla y - 2\omega \nabla y \cdot v}{|\nabla y|^2 + \gamma}, z \right) + 2 \left(\frac{(\omega \nabla y \cdot y) \nabla y \cdot \nabla v}{(|\nabla y|^2 + \gamma)^2}, z \right) \\ &\leq \left\| \frac{|\nabla y|^2}{(|\nabla y|^2 + \gamma)} \right\|_{L^\infty} \|\chi\|_{L^2} \|z\|_{L^2} + \left\| \frac{|\nabla y|}{(|\nabla y|^2 + \gamma)} \right\|_{L^\infty} \|\nabla v\|_{L^2} \|z\|_{L^q} \|\omega\|_{L^{\frac{2q}{q-2}}} \\ &\quad + \left\| \frac{|\nabla y|^3}{(|\nabla y|^2 + \gamma)^2} \right\|_{L^\infty} \|\nabla v\|_{L^2} \|\omega\|_{L^{\frac{2q}{q-2}}} \|z\|_{L^q} \\ &\leq C\gamma^{-\frac{1}{2}} \|\omega\|_{L^{\frac{2q}{q-2}}} (\|\chi\|_{L^2} + \|v\|_{H^1}) \|z\|_{H^1}. \end{aligned}$$

It then follows that

$$\|F'[\omega, y](\chi, v)\|_{H^{-1}} \leq C\|(\omega, y)\|_{X \times Y}(\|\chi\|_{L^2} + \|v\|_{H^1}),$$

and thus, assumptions [B3]–[B4] hold with

$$(6.93) \quad \|(\Pi^h \sigma^\varepsilon - \tau \sigma^\varepsilon, \mathcal{I}^h u^\varepsilon - \tau u^\varepsilon)\|_{X \times Y} = \gamma^{-\frac{1}{2}} \|\Pi^h \sigma^\varepsilon - \tau \sigma^\varepsilon\|_{L^{\frac{2q}{q-1}}} = \tilde{K}_3(\varepsilon).$$

Next, for $(\mu_h, v_h) \in W_\varepsilon^h \times Q_g^h$ with $\|(\Pi^h \sigma^\varepsilon - \mu_h, \mathcal{I}^h u^\varepsilon - v_h)\|_\varepsilon \leq \delta \in (0, \frac{1}{2})$ and $(\kappa_h, z_h) \in W^h \times Q^h$

$$(6.94) \quad \begin{aligned} & (F'[\sigma^\varepsilon, u^\varepsilon] - F'[\mu_h, v_h])(\kappa_h, z_h) \\ &= \frac{\kappa_h \nabla v_h \cdot \nabla v_h}{|\nabla v_h|^2 + \gamma} - \frac{\kappa_h \nabla u^\varepsilon \cdot \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma} + 2 \left(\frac{\mu_h \nabla v_h}{|\nabla v_h|^2 + \gamma} - \frac{\sigma^\varepsilon \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma} \right) \cdot \nabla z_h \\ & \quad + 2 \left(\frac{(\sigma^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon) \nabla u^\varepsilon \cdot \nabla z_h}{(|\nabla u^\varepsilon|^2 + \gamma)^2} - \frac{(\mu_h \nabla v_h \cdot \nabla v_h) \nabla v_h \cdot \nabla z_h}{(|\nabla v_h|^2 + \gamma)^2} \right). \end{aligned}$$

To bound the first term in (6.94), we add and subtract terms to deduce

$$\begin{aligned} & \frac{\kappa_h \nabla v_h \cdot \nabla v_h}{|\nabla v_h|^2 + \gamma} - \frac{\kappa_h \nabla u^\varepsilon \cdot \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma} \\ &= \frac{\kappa_h \nabla v_h \cdot \nabla v_h - \kappa_h \nabla u^\varepsilon \cdot \nabla u^\varepsilon}{|\nabla v_h|^2 + \gamma} + \kappa_h \nabla u^\varepsilon \cdot \nabla u^\varepsilon \left(\frac{1}{|\nabla v_h|^2 + \gamma} - \frac{1}{|\nabla u^\varepsilon|^2 + \gamma} \right) \\ &= \frac{\kappa_h (\nabla v_h + \nabla u^\varepsilon) \cdot (\nabla v_h - \nabla u^\varepsilon)}{|\nabla v_h|^2 + \gamma} + \kappa_h \nabla u^\varepsilon \cdot \nabla u^\varepsilon \left(\frac{(\nabla u^\varepsilon - \nabla v_h)(\nabla u^\varepsilon + \nabla v_h)}{(|\nabla v_h|^2 + \gamma)(|\nabla u^\varepsilon|^2 + \gamma)} \right), \end{aligned}$$

and therefore by the inverse inequality,

$$(6.95) \quad \begin{aligned} & \left\| \frac{\kappa_h \nabla v_h \cdot \nabla v_h}{|\nabla v_h|^2 + \gamma} - \frac{\kappa_h \nabla u^\varepsilon \cdot \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma} \right\|_{L^1} \\ & \leq C \left(\|\nabla v_h + \nabla u^\varepsilon\|_{L^2} \|\nabla v_h - \nabla u^\varepsilon\|_{L^2} \right. \\ & \quad \left. + \|\nabla u^\varepsilon\|_{L^\infty}^2 \|\nabla u^\varepsilon - \nabla v_h\|_{L^2} \|\nabla u^\varepsilon + \nabla v_h\|_{L^2} \right) \|\kappa_h\|_{L^\infty} \\ & \leq Ch^{-1} \|u^\varepsilon\|_{H^1} \|\nabla u^\varepsilon\|_{L^\infty}^2 \|u^\varepsilon - v_h\|_{H^1} \|\kappa_h\|_{L^2}. \end{aligned}$$

Using a similar technique to bound the second term in (6.94), we first write

$$\begin{aligned} & \left(\frac{\mu_h \nabla v_h}{|\nabla v_h|^2 + \gamma} - \frac{\sigma^\varepsilon \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma} \right) \cdot \nabla z_h \\ &= \left(\frac{\mu_h \nabla v_h - \sigma^\varepsilon \nabla u^\varepsilon}{|\nabla v_h|^2 + \gamma} + \sigma^\varepsilon \nabla u^\varepsilon \left(\frac{|\nabla u^\varepsilon|^2 - |\nabla v_h|^2}{(|\nabla v_h|^2 + \gamma)(|\nabla u^\varepsilon|^2 + \gamma)} \right) \right) \cdot \nabla z_h \\ &= \left(\frac{(\mu_h - \sigma^\varepsilon) \nabla v_h + \sigma^\varepsilon (\nabla v_h - \nabla u^\varepsilon)}{|\nabla v_h|^2 + \gamma} + \sigma^\varepsilon \nabla u^\varepsilon \left(\frac{(\nabla u^\varepsilon + \nabla v_h)(\nabla u^\varepsilon - \nabla v_h)}{(|\nabla v_h|^2 + \gamma)(|\nabla u^\varepsilon|^2 + \gamma)} \right) \right) \cdot \nabla z_h. \end{aligned}$$

It then follows that

$$\begin{aligned}
(6.96) \quad & \left\| \left(\frac{\mu_h \nabla v_h}{|\nabla v_h|^2 + \gamma} - \frac{\sigma^\varepsilon \nabla u^\varepsilon}{|\nabla u^\varepsilon|^2 + \gamma} \right) \cdot \nabla z_h \right\|_{L^1} \\
& \leq C \left(\|\mu_h - \sigma^\varepsilon\|_{L^2} \|\nabla v_h\|_{L^\infty} \|\nabla z_h\|_{L^2} \right. \\
& \quad \left. + \|\sigma^\varepsilon\|_{L^\infty} \|\nabla v_h - \nabla u^\varepsilon\|_{L^2} \|\nabla z_h\|_{L^2} \right. \\
& \quad \left. + \|\sigma^\varepsilon\|_{L^\infty} \|\nabla u^\varepsilon\|_{L^\infty} \|\nabla u^\varepsilon + \nabla v_h\|_{L^2} \|\nabla u^\varepsilon - \nabla v_h\|_{L^2} \|\nabla z_h\|_{L^\infty} \right) \\
& \leq C \left(h^{-1} \|\mu_h - \sigma^\varepsilon\|_{L^2} \|u^\varepsilon\|_{H^1} + \|\sigma^\varepsilon\|_{L^\infty} \|u^\varepsilon - v_h\|_{H^1} \right. \\
& \quad \left. + h^{-1} \|\sigma^\varepsilon\|_{L^\infty} \|\nabla u^\varepsilon\|_{L^\infty} \|u^\varepsilon\|_{H^1} \|u^\varepsilon - v_h\|_{H^1} \right) \|z_h\|_{H^1} \\
& \leq Ch^{-1} \|\sigma^\varepsilon\|_{L^\infty} \|\nabla u^\varepsilon\|_{L^\infty} \|u^\varepsilon\|_{H^1} (\|\sigma^\varepsilon - \mu_h\|_{L^2} + \|u^\varepsilon - v_h\|_{H^1}) \|z_h\|_{H^1}.
\end{aligned}$$

Next, we write

$$\begin{aligned}
& \frac{(\sigma^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon) \nabla u^\varepsilon \cdot \nabla z_h}{(|\nabla u^\varepsilon|^2 + \gamma)^2} - \frac{(\mu_h \nabla v_h \cdot \nabla v_h) \nabla v_h \cdot \nabla z_h}{(|\nabla v_h|^2 + \gamma)^2} \\
& = \frac{(\sigma^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon) \nabla u^\varepsilon \cdot \nabla z_h - (\mu_h \nabla v_h \cdot \nabla v_h) \nabla v_h \cdot \nabla z_h}{(|\nabla v_h|^2 + \gamma)^2} \\
& \quad + (\sigma^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon) \nabla u^\varepsilon \cdot \nabla z_h \left(\frac{1}{(|\nabla u^\varepsilon|^2 + \gamma)^2} - \frac{1}{(|\nabla v_h|^2 + \gamma)^2} \right)
\end{aligned}$$

Noting

$$\begin{aligned}
& (\sigma^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon) \nabla u^\varepsilon \cdot \nabla z_h - (\mu_h \nabla v_h \cdot \nabla v_h) \nabla v_h \cdot \nabla z_h \\
& = ((\sigma^\varepsilon - \mu_h) \nabla u^\varepsilon \cdot \nabla u^\varepsilon) (\nabla u^\varepsilon \cdot \nabla z_h) + (\mu_h (\nabla u^\varepsilon - \nabla v_h) \cdot (\nabla u^\varepsilon + \nabla v_h)) (\nabla u^\varepsilon \cdot \nabla z_h) \\
& \quad + (\mu_h \nabla v_h \cdot \nabla v_h) (\nabla u^\varepsilon - \nabla v_h) \cdot \nabla z_h,
\end{aligned}$$

we conclude

$$\begin{aligned}
(6.97) \quad & \left\| \frac{(\sigma^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon) \nabla u^\varepsilon \cdot \nabla z_h - (\mu_h \nabla v_h \cdot \nabla v_h) \nabla v_h \cdot \nabla z_h}{(|\nabla v_h|^2 + \gamma)^2} \right\|_{L^1} \\
& \leq C \left(\|\sigma^\varepsilon - \mu_h\|_{L^2} \|\nabla u^\varepsilon\|_{L^\infty}^3 \|\nabla z_h\|_{L^2} \right. \\
& \quad \left. + \|\mu_h\|_{L^2} \|\nabla u^\varepsilon - \nabla v_h\|_{L^2} \|\nabla u^\varepsilon + \nabla v_h\|_{L^\infty} \|\nabla u^\varepsilon\|_{L^\infty} \|\nabla z_h\|_{L^\infty} \right. \\
& \quad \left. + \|\mu_h\|_{L^2} \|\nabla v_h\|_{L^\infty}^2 \|\nabla u^\varepsilon - \nabla v_h\|_{L^2} \|\nabla z_h\|_{L^\infty} \right) \\
& \leq C \left(\|\nabla u^\varepsilon\|_{L^\infty}^3 + h^{-2} \|\sigma^\varepsilon\|_{L^2} \|u^\varepsilon\|_{H^1} \|\nabla u^\varepsilon\|_{L^\infty} \right. \\
& \quad \left. + h^{-3} \|\sigma^\varepsilon\|_{L^2} \|u^\varepsilon\|_{H^1}^2 \right) \|u^\varepsilon - v_h\|_{H^1} \|z_h\|_{H^1}.
\end{aligned}$$

We also have

$$\begin{aligned}
& (\sigma^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon) \nabla u^\varepsilon \cdot \nabla z_h \left(\frac{1}{(|\nabla u^\varepsilon|^2 + \gamma)^2} - \frac{1}{(|\nabla v_h|^2 + \gamma)^2} \right) \\
& = (\sigma^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon) \nabla u^\varepsilon \cdot \nabla z_h \left(\frac{(|\nabla v_h|^2 + |\nabla u^\varepsilon|^2 + \gamma)(\nabla v_h - \nabla u^\varepsilon)(\nabla v_h + \nabla u^\varepsilon)}{(|\nabla v_h|^2 + \gamma)^2 (|\nabla u^\varepsilon|^2 + \gamma)^2} \right),
\end{aligned}$$

and therefore,

$$\begin{aligned}
(6.98) \quad & \left\| (\sigma^\varepsilon \nabla u^\varepsilon \cdot \nabla u^\varepsilon) \nabla u^\varepsilon \cdot \nabla z_h \left(\frac{1}{(|\nabla u^\varepsilon|^2 + \gamma)^2} - \frac{1}{(|\nabla v_h|^2 + \gamma)^2} \right) \right\|_{L^1} \\
& \leq C \left(\|\sigma^\varepsilon\|_{L^\infty} \|\nabla u^\varepsilon\|_{L^\infty}^3 \|\nabla z_h\|_{L^\infty} \|\nabla v_h\|^2 \right. \\
& \quad \left. + \|\nabla u^\varepsilon\|^2 + \gamma\|_{L^4} \|\nabla v_h - \nabla u^\varepsilon\|_{L^2} \|\nabla v_h + \nabla u^\varepsilon\|_{L^4} \right) \\
& \leq Ch^{-3} \|\sigma^\varepsilon\|_{L^\infty} \|\nabla u^\varepsilon\|_{L^\infty}^3 \|u^\varepsilon\|_{H^2}^3 \|u^\varepsilon - v_h\|_{H^1} \|z_h\|_{H^1}.
\end{aligned}$$

Combining (6.95)–(6.98), we have

$$\begin{aligned}
\|F'[\sigma^\varepsilon, u^\varepsilon] - F'[\mu_h, v_h]\|_{L^1} & \leq C \left(h^{-3} \|\sigma^\varepsilon\|_{L^\infty} \|\nabla u^\varepsilon\|_{L^\infty}^3 \|u^\varepsilon\|_{H^2}^3 \right) \\
& \quad \times (\|\sigma^\varepsilon - \mu_h\|_{L^2} + \|u^\varepsilon - v_h\|_{H^1}) (\|\kappa_h\|_{L^2} + \|z_h\|_{H^1}).
\end{aligned}$$

It then follows from the inverse inequality, that

$$\begin{aligned}
& \sup_{w \in Q^h} \frac{\langle (F'[\sigma^\varepsilon, u^\varepsilon] - F'[\mu_h, v_h]) (\kappa_h, z_h), w_h \rangle}{\|w_h\|_{H^1}} \\
& \leq C |\log h|^{\frac{1}{2}} \left(h^{-3} \|\sigma^\varepsilon\|_{L^\infty} \|\nabla u^\varepsilon\|_{L^\infty}^3 \|u^\varepsilon\|_{H^2}^3 \right) \\
& \quad \times (\|\sigma^\varepsilon - \mu_h\|_{L^2} + \|u^\varepsilon - v_h\|_{H^1}) (\|\kappa_h\|_{L^2} + \|z_h\|_{H^1}),
\end{aligned}$$

and therefore condition [B5] holds with

$$R(h) = C |\log h|^{\frac{1}{2}} \left(h^{-3} \|\sigma^\varepsilon\|_{L^\infty} \|\nabla u^\varepsilon\|_{L^\infty}^3 \|u^\varepsilon\|_{H^2}^3 \right).$$

Finally, we confirm assumption [B6]. First, we note that

$$\frac{\partial F(\sigma^\varepsilon, u^\varepsilon)}{\partial r_{ij}} = \frac{\frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial u^\varepsilon}{\partial x_j}}{|\nabla u^\varepsilon|^2 + \gamma},$$

and

$$\frac{\partial}{\partial x_k} \left(\frac{\partial F(\sigma^\varepsilon, u^\varepsilon)}{\partial r_{ij}} \right) = \frac{\frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_k} \frac{\partial u^\varepsilon}{\partial x_j} + \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial^2 u^\varepsilon}{\partial x_j \partial x_k}}{|\nabla u^\varepsilon|^2 + \gamma} - 2 \frac{\frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial u^\varepsilon}{\partial x_j} (D^2 u^\varepsilon \nabla u^\varepsilon)_k}{(|\nabla u^\varepsilon|^2 + \gamma)^2},$$

and therefore by (6.76)

$$\begin{aligned}
& \max_{1 \leq i, j \leq 2} \left\| \frac{\partial F(\sigma^\varepsilon, u^\varepsilon)}{\partial r_{ij}} \right\|_{L^\infty} \leq C, \\
& \max_{1 \leq i, j \leq 2} \left\| \frac{\partial F(\sigma^\varepsilon, u^\varepsilon)}{\partial r_{ij}} \right\|_{W^{1, \frac{6}{5}}} \leq C \left(\|u^\varepsilon\|_{W^{2, \frac{6}{5}}} + \|\nabla u^\varepsilon\|_{L^\infty} \|D^2 u^\varepsilon\|_{L^{\frac{6}{5}}} \right) \\
& \leq C \|\nabla u^\varepsilon\|_{L^\infty} \|u^\varepsilon\|_{W^{2, \frac{6}{5}}},
\end{aligned}$$

and therefore by Proposition 5.4, condition [B6] holds with

$$(6.99) \quad \alpha = 1, \quad K_G = C \|\nabla u^\varepsilon\|_{L^\infty} \|u^\varepsilon\|_{W^{2, \frac{6}{5}}}.$$

Finally, we apply Theorem 5.15 to obtain existence and uniqueness of a solution $(\tilde{\sigma}_h^\varepsilon, u_h^\varepsilon)$ to the mixed finite element method (6.68)–(6.70) as well as the estimates (6.90)–(6.91). \square

6.3.3. Numerical experiments and rates of convergence.

Test 6.3.1. In this test, we numerically solve the infinity-Laplacian equation using the Argyris element of degree $k = 5$ for fixed $h = 0.015$ while varying ε . The purpose of these experiments is to estimate the rate of convergence of $\|u - u^\varepsilon\|$ in various norms, where u is the viscosity solution of (6.65)–(6.66). To this end, we solve the following finite element method (compare to (6.73)): find $u_h^\varepsilon \in V_g^h$ such that

(6.100)

$$\varepsilon(\Delta u_h^\varepsilon, \Delta v_h) - \left(\frac{\tilde{\Delta}_\infty u_h^\varepsilon}{|\nabla u_h^\varepsilon|^2 + \gamma}, v_h \right) = (f, v_h) + \left\langle \varepsilon^2, \frac{\partial v_h}{\partial \nu} \right\rangle_{\partial \Omega} \quad \forall v_h \in V_0^h.$$

We set $\Omega = (-0.5, 0.5)^2$, $\gamma = \varepsilon^2$, and use the following two test functions:

$$(a) \quad u = x_1^{4/3} - x_2^{4/3}, \quad f = 0,$$

$$(b) \quad u = x_1^2 + x_2^2, \quad f = \frac{8(x^2 + y^2)}{4x^2 + 4y^2 + \gamma}.$$

We note that the second test function is smooth, but the first does not belong to $C^2(\Omega)$ since its second derivatives have singularities at $x_1 = 0$ and $x_2 = 0$. After computing the solution for different ε -values, we list the errors in Table 7 with their estimated rate of convergence and plot the results in Figure 15. The numerical experiments indicate the following rates of convergence as $\varepsilon \rightarrow 0^+$:

$$\|u - u_h^\varepsilon\|_{L^2} \approx O\left(\varepsilon^{\frac{2}{3}}\right), \quad \|u - u_h^\varepsilon\|_{H^1} \approx O\left(\varepsilon^{\frac{1}{3}}\right), \quad \|u - u_h^\varepsilon\|_{H^2} \approx O\left(\varepsilon^{\frac{1}{6}}\right).$$

Since we have fixed h small, we expect that $\|u - u^\varepsilon\|$ has similar rates of convergence.

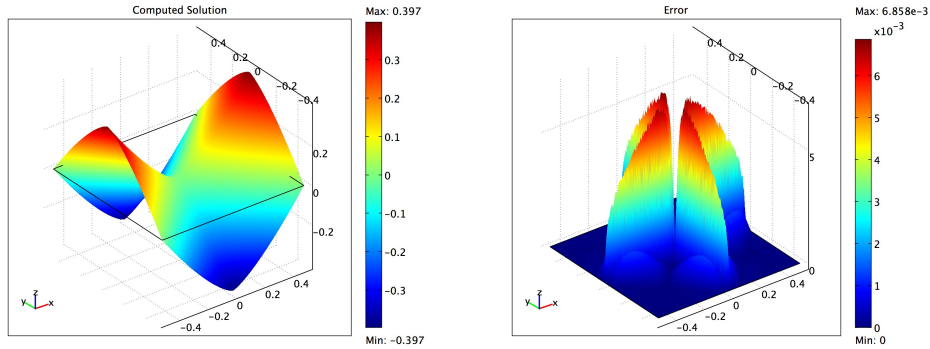


FIGURE 14. Test 6.3.1a. Computed solution (left) and its error (right) with $\varepsilon = 0.001$ and $h = 0.015$.

Test 6.3.2. For our last test, we verify the theoretical results derived in Section 6.3.2. To this end, we solve the following problem: find $(\tilde{\sigma}_h^\varepsilon, u_h^\varepsilon) \in \tilde{W}_{\phi^\varepsilon}^h$ such that

$$(6.101) \quad (\tilde{\sigma}_h^\varepsilon, \mu_h) + \tilde{b}(\mu_h, u_h^\varepsilon) = G(\mu_h) \quad \forall \mu_h \in W_0^h,$$

$$(6.102) \quad \tilde{b}(\tilde{\sigma}_h^\varepsilon, v_h) - \varepsilon^{-1} \tilde{c}(\tilde{\sigma}_h^\varepsilon, u_h^\varepsilon, v_h) = (f^\varepsilon, v_h) \quad \forall v_h \in Q_0^h,$$

where

$$\tilde{W}_{\phi^\varepsilon}^h := \{\mu_h \in W^h; \mu_h \nu \cdot \nu|_{\partial \Omega} = \phi^\varepsilon + \tau g\}.$$

TABLE 7. Test 6.3.1. Error of $\|u - u_h^\varepsilon\|$ w.r.t ε ($h = 0.015$)

	ε	$\ u - u_h^\varepsilon\ _{L^2}(\text{rate})$	$\ u - u_h^\varepsilon\ _{H^1}(\text{rate})$	$\ u - u_h^\varepsilon\ _{H^2}(\text{rate})$
Test 6.3.1a	1.0E-03	2.15E-03(—)	3.90E-02(—)	—
	5.0E-04	1.52E-03(0.50)	3.29E-02(0.25)	—
	2.5E-04	1.06E-03(0.52)	2.75E-02(0.26)	—
	1.0E-04	6.54E-04(0.53)	2.15E-02(0.27)	—
	5.0E-05	4.51E-04(0.54)	1.77E-02(0.28)	—
	2.5E-05	3.09E-04(0.54)	1.45E-02(0.29)	—
	1.0E-05	1.88E-04(0.55)	1.10E-02(0.30)	—
Test 6.3.1b	1.0E-03	1.01E-02(—)	7.20E-02(—)	1.36E+00(—)
	5.0E-04	6.02E-03(0.75)	5.10E-02(0.50)	1.21E+00(0.16)
	2.5E-04	3.61E-03(0.74)	3.70E-02(0.46)	1.08E+00(0.16)
	1.0E-04	1.86E-03(0.72)	2.50E-02(0.43)	9.36E-01(0.15)
	5.0E-05	1.15E-03(0.70)	1.89E-02(0.41)	8.44E-01(0.15)
	2.5E-05	7.14E-04(0.68)	1.44E-02(0.39)	7.63E-01(0.15)
	1.0E-05	3.87E-04(0.67)	1.01E-02(0.39)	6.70E-01(0.14)

We use the following test function:

$$u^\varepsilon = \cos(x_1) - \cos(x_2), \quad \phi^\varepsilon = \nu_2^2 \cos(x_2) - \nu_1^2 \cos(x_1),$$

$$f^\varepsilon = \varepsilon(\cos(x_1) - \cos(x_2)) + \frac{\cos(x_1) \sin^2(x_1) - \cos(x_2) \sin^2(x_2)}{\sin^2(x_1) + \sin^2(x_2) + \gamma}.$$

We compute (6.101)–(6.102) for fixed $\varepsilon = 0.01$, while varying h with $\Omega = (-0.5, 0.5)^2$ and $\gamma = \varepsilon^2 = 1\text{E-}4$. We list the error of the computed solution in Table 8 for both $\tau = 0$ and $\tau = 1$. As expected, for the case $\tau = 1$, we observe the following rates of convergence:

$$\|u^\varepsilon - u_h^\varepsilon\|_{L^2} = O(h^3), \quad \|u^\varepsilon - u_h^\varepsilon\|_{H^1} = O(h^2), \quad \|\tilde{\sigma}^\varepsilon - \tilde{\sigma}_h^\varepsilon\|_{L^2} = O(h).$$

We also observe that the same rates of convergence appear to hold for the case $\tau = 0$, although our theoretical results of Section 6.3.2 do not cover this case.

TABLE 8. Test 6.3.2. Error of $\|u^\varepsilon - u_h^\varepsilon\|$ w.r.t h ($\varepsilon = 0.01$)

	h	$\ u^\varepsilon - u_h^\varepsilon\ _{L^2}(\text{rate})$	$\ u^\varepsilon - u_h^\varepsilon\ _{H^1}(\text{rate})$	$\ \tilde{\sigma}^\varepsilon - \tilde{\sigma}_h^\varepsilon\ _{L^2}(\text{rate})$
$\tau = 0$	2.0E-01	8.74E-06(—)	3.92E-04(—)	9.93E-03(—)
	1.0E-01	1.14E-06(2.94)	1.03E-04(1.93)	4.11E-03(1.27)
	5.0E-02	1.38E-07(3.05)	2.53E-05(2.02)	1.54E-03(1.41)
	2.5E-02	1.67E-08(3.05)	6.22E-06(2.03)	5.08E-04(1.60)
	1.0E-02	1.33E-09(2.76)	9.98E-07(2.00)	1.29E-04(1.49)
$\tau = 1$	2.0E-01	8.74E-06(—)	3.92E-04(—)	9.93E-03(—)
	1.0E-01	1.14E-06(2.94)	1.03E-04(1.93)	4.11E-03(1.27)
	5.0E-02	1.38E-07(3.05)	2.53E-05(2.02)	1.54E-03(1.41)
	2.5E-02	1.66E-08(3.05)	6.22E-06(2.03)	5.08E-04(1.60)
	1.0E-02	1.22E-09(2.85)	9.98E-07(2.00)	1.29E-04(1.49)

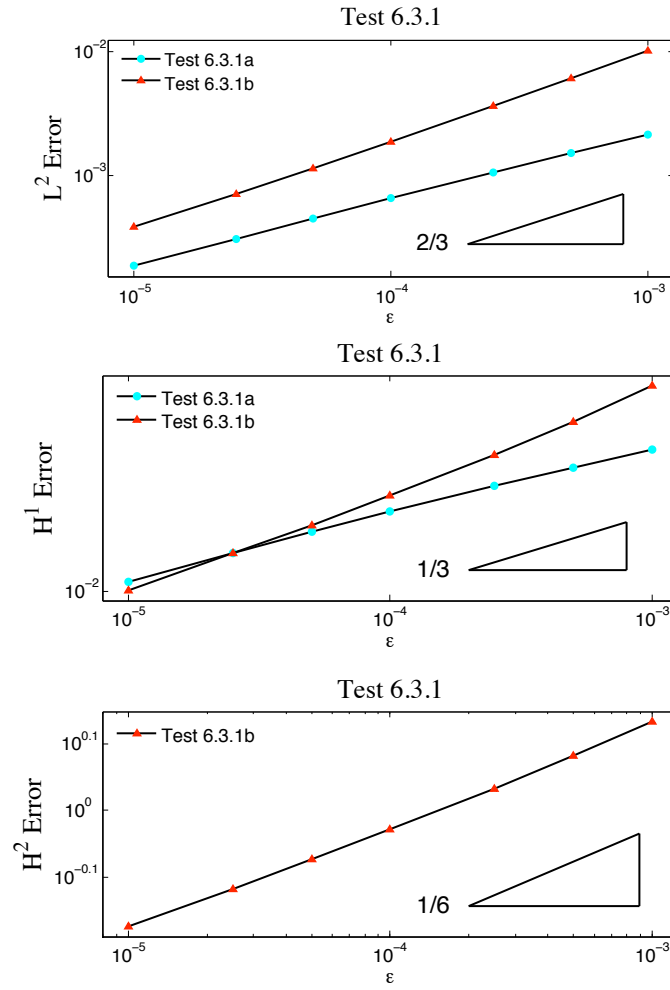


FIGURE 15. Test 6.3.1. Error $\|u - u_h^\varepsilon\|_{L^2}$ (top), $\|u - u_h^\varepsilon\|_{H^1}$ (middle), and $\|u - u_h^\varepsilon\|_{H^2}$ (bottom) w.r.t. ε ($h = 0.015$).

CHAPTER 7

Concluding Comments

In this final chapter, we give some concluding comments about the vanishing moment method and its finite element and mixed finite element approximations for fully nonlinear second order PDEs. In particular, we point out some main issues accompanying with the methodology.

We recall that the vanishing moment method and the notion of moment solutions are exactly in the same spirit as the vanishing viscosity method and the original notion of viscosity solutions proposed by M. Crandall and P. L. Lions in [24] for the Hamilton-Jacobi equations, which is based on the idea of approximating a fully nonlinear PDE by a family of quasilinear higher order PDEs. The vanishing moment method then allows one to reliably compute the viscosity solutions of fully nonlinear second order PDEs, in particular, using Galerkin-type methods and existing numerical methods and computer software (with slight modifications), a task which had been impracticable before. As a by-product, the vanishing moment method reveals some insights for the understanding of viscosity solutions, and the notion of moment solutions might also provide a logical and natural generalization/extension for the notion of viscosity solution, especially, in the cases where there is no theory or the existing viscosity solution theory fails (e.g. the Monge-Ampère equations of hyperbolic type [20] and systems of fully nonlinear second order PDEs.)

7.1. Boundary layers

As pointed out in Chapter 2, in order to approximate a second order PDE by a quasilinear fourth order PDE, we must impose an extra boundary condition such as those given in (2.11). Because the extra boundary condition is artificial, it is expected that a “boundary layer” ought be introduced in an ε -neighborhood of $\partial\Omega$. For example, in the case that $\Delta u^\varepsilon = \varepsilon$ is used as the extra boundary condition on $\partial\Omega$, and since we do not know a priori the true value Δu on $\partial\Omega$ (note that Δu may not even exist if the viscosity solution u is not differentiable), Δu^ε and Δu take different values on $\partial\Omega$ in general, and the discrepancy between Δu^ε and Δu could be large although this can only occur in a very small region (i.e., an ε -neighborhood of $\partial\Omega$).

Since the convergence of u^ε to u as $\varepsilon \searrow 0^+$ is only expected and proved in low order norms (cf. Chapters 2 and 3), the error $\Delta u^\varepsilon - \Delta u$ in an ε -neighborhood of $\partial\Omega$ does not cause any problem for the convergence. Our numerical experiments do confirm this conclusion. Moreover, as expected, our numerical experiments also confirm that $\|u^\varepsilon - u\|_{H^2}$ does not converge in general (cf. Test 6.1.3). On the other hand, a closer look at the error of computed solution in Figure 5 shows that the error is concentrated in an ε -neighborhood of $\partial\Omega$ and at the singularity of the solution u .

To improve the accuracy and efficiency of the vanishing moment method, we propose the following simple *iterative surgical strategy*, which consists of three steps.

Step 1: Solve numerically (2.9)–(2.11)₁ as before for a fixed (small) $\varepsilon > 0$.

Step 2: Find Δu_h^ε on the inner boundary of the ε -neighborhood of $\partial\Omega$, and extend the function to $\partial\Omega$ by any (convenient) method. We denote the extended function by c_ε .

Step 3: Solve numerically (2.9)–(2.11)₁ again with $\Delta u^\varepsilon|_{\partial\Omega} = \varepsilon$ being replaced by $\Delta u^\varepsilon|_{\partial\Omega} = c_\varepsilon$.

Remark 7.1. (a) c_ε can be obtained by an interpolation technique, or by doing a ray tracing along the normal on $\partial\Omega$, or simply by letting c_ε be the maximum value (a constant) of Δu_h^ε on the inner boundary of the ε -neighborhood of $\partial\Omega$.

(b) Clearly, *Step 2* and *Step 3* can be repeated, although one iteration is often sufficient in practice (see numerical experiment below).

(c) The above *iterative surgical strategy* is a “predictor-corrector” type strategy, where the prediction and correction are done on $\Delta u_h^\varepsilon|_{\partial\Omega}$.

(d) To make the algorithm more efficient, the solution computed in Step 1 can be used as an initial guess for the nonlinear solver in Step 3.

As a numerical example for the *iterative surgical strategy*, we solve the Monge-Ampère equation using the conforming finite element method developed and analyzed in Section 6.1.1, that is, we numerically solve (2.9) with $F(D^2u^\varepsilon, \nabla u^\varepsilon, u^\varepsilon, x) = f(x) - \det(D^2u^\varepsilon)$ using the finite element method (6.6). Here, we use fifth degree Argyris elements to construct the finite element space, and set $\Omega = (0, 1)^2$, $f = (1 + x_1^2 + x_2^2)e^{x_1^2 + x_2^2}$, so that the exact solution is $u = e^{(x_1^2 + x_2^2)/2}$.

In Step 2, we extend Δu_h^ε in the neighborhood of $\partial\Omega$, to $\partial\Omega$ by linear interpolation to construct c_ε . After performing Steps 1–3, we repeat Steps 2 and 3 four more times to determine whether repeated iterations make a significant impact on the error. We use the parameters $\varepsilon = 0.01$ and $h = 0.01$ for all computations.

After computing the solutions in Step 1 and 3, we record the errors in Tables 1–2. We also plot the cross-section of the computed Laplacian Δu_h^ε at $x_2 = 0.8$ in Figure 1 after each iteration. Tables 1–2 clearly indicate the iterative surgical strategy decreases the error at each step. In fact, the error in every norm is decreased by nearly a factor of ten by performing Steps 1–3 just once. However, the error decreases only modestly after repeated iterations and has no impact on the L^2 and H^1 errors after two iterations. Figure 1 also indicates that the boundary layer is greatly reduced after the first iteration, and improves modestly after each subsequent iteration.

iteration #	$\ u - u_h^\varepsilon\ _{L^2}$	$\ u - u_h^\varepsilon\ _{H^1}$	$\ u - u_h^\varepsilon\ _{H^2}$
0	1.48E-02	1.00E-01	1.79E+00
1	1.88E-03	2.11E-02	4.63E-01
2	1.51E-03	1.23E-02	2.53E-01
3	2.15E-03	1.18E-02	1.77E-01
4	2.51E-03	1.24E-02	1.42E-01

TABLE 1. Errors of $u - u_h^\varepsilon$ using the iterative surgical strategy ($\varepsilon = 0.01, h = 0.01$).

iteration #	$\ u - u_h^\varepsilon\ _{L^\infty}$	$\ u - u_h^\varepsilon\ _{W^{1,\infty}}$	$\ u - u_h^\varepsilon\ _{W^{2,\infty}}$
0	2.02E-02	4.25E-01	2.93E+01
1	3.94E-03	9.50E-02	6.14E+00
2	3.52E-03	5.06E-02	3.83E+00
3	4.66E-03	3.93E-02	2.79E+00
4	5.21E-03	3.31E-02	2.50E+00

TABLE 2. Pointwise errors of $u - u_h^\varepsilon$ using the iterative surgical strategy ($\varepsilon = 0.01, h = 0.01$).

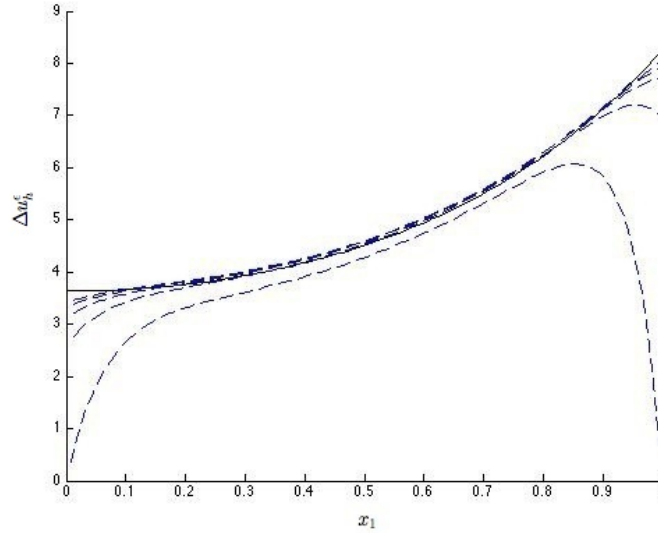


FIGURE 1. Cross-section plot of Δu_h^ε at $x_2 = 0.8$. Black solid line is exact solution, and dotted blue lines are the computed solutions for iterations 0, 1, 2, 3, and 4.

7.2. Nonlinear solvers

After problem (2.9)–(2.11)₁ is discretized, we obtain the (strong) nonlinear algebraic system (4.3) or (5.8)–(5.9) or (5.57)–(5.58) to solve. To this end, one has to use one or another iterative methods to do the job. In all numerical experiments given in Chapter 6, we use the ILU preconditioned Newton iterative method as our nonlinear solver. Since Newton’s method often requires an accurate starting value to ensure convergence, hence generating a good starting value for Newton’s method is also an important issue here. So far we have used two strategies for the purpose in our numerical experiments in [37, 38, 39] and in Chapter 6. The first strategy is to use a fixed point iteration to generate a starting value for Newton’s method. However, this strategy may not always work although its success rate is pretty high. The second strategy, which is more involved, is the following “multi-resolution” or “homotopy” strategy: first compute a numerical solution using a relatively large ε , then use the computed solution as a starting value for the Newton method at

a finer resolution ε . The process may need to be iterated in ε for more than one step. Our experiences tell that 1 – 3 steps should be enough to generate a good starting value for Newton’s method at the finest resolution ε at which one wants to compute a solution.

It is expected that for 3-d simulations and for time-dependent fully nonlinear PDEs (see Section 7.3 below), more efficient fast solvers are required. It is well-known that the key to this is to use better preconditioners for the linear problem inside each Newton iteration because solving (large) linear systems inside each Newton iteration costs most of the total CPU time for executing the Newton’s method. One plausible approach, which will be pursued in a future work, is to use more sophisticated multigrid or Schwarz (or domain decomposition) preconditioners (cf. [74]) to replace the ILU preconditioner. With help of the better preconditioners, Krylov subspace methods [71] can be employed as the linear solver inside each Newton iteration. Put all pieces together, we arrive at a global nonlinear iterative solver which can be called the Newton-Schwarz/Multigrid-Krylov method (cf. [52]).

7.3. Open problems

As the vanishing moment method was introduced very recently, there are many open questions concerning with the method. The foremost one is to generalize the convergence results of Chapter 3 to the general problem (2.9)–(2.11) under some reasonable structure conditions on the nonlinear differential operator F . The convergence rate is probably hard to get unless the viscosity solution of the limiting problem (2.7)–(2.8) is sufficiently regular (cf. Theorem 3.19).

Another interesting but completely open problem is to develop a vanishing moment method for fully nonlinear second order parabolic PDEs. Unlike the situation for quasilinear PDEs, going from fully nonlinear second order elliptic PDEs to fully nonlinear second order parabolic PDEs is far from straightforward. One reason for this is that there are several different legitimate parabolic generalizations for equation (2.7) (cf. [55, 75, 76]). Two best known fully nonlinear second order parabolic PDEs are

$$(7.1) \quad F(D^2u, \nabla u, u, x, t) - u_t = 0,$$

$$(7.2) \quad -u_t \det(D^2u) = f(\nabla u, u, x, t) \geq 0.$$

Extensive viscosity solution theories have been developed for both equations (cf. [45, 55, 75, 76] and the references therein). However, to the best of our knowledge, no numerical work has been reported for these equations in the literature.

Formulation of the vanishing moment method for (7.1) is straightforward (see [37]). By adopting the method of lines approach, generalizations of the finite element and mixed finite element methods of Chapter 4 and 5 should be standard. However, the convergence analysis of any implicit scheme is expected to be hard, in particular, establishing error estimates which depend on ε^{-1} *polynomially* instead of *exponentially* will be very challenging. Furthermore, we note that numerically solving equation (7.2) using the vanishing moment method is expected to be difficult. In fact, it is not clear how to formulate the method for (7.2).

Finally, another interesting open question is to explore the feasibility of extending the notion of moment solutions and the vanishing moment method to

degenerate, non-elliptic, and systems of fully nonlinear second order PDEs (cf. [11, 17, 40, 20, 58]).

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